"In mathematics you don't understand things. You just get used to them."

John von Neumann

9
Double Integrals

General facts

• for a complete introduction of this topic have a look in the book [1] of J. Stewart (I sent you the chapter about the double integral a few weeks ago)

• in the sequel we'll discuss double integration from a computational point of view, no special attention will be given to the existence conditions or other possible aspects

Fact no 1: *Integration over rectangles is easy*

Theorem of Fubini

Let f be a continuous function over the rectangle

$$R = \{(x,y): \ a \le x \le b, \quad c \le y \le d\},$$

then the following identities hold

$$\iint_{[a,b]\times[c,d]} f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Consequences

For a function with separated variables $f(x,y) = f(x)g(y)$ one gets

$$\iint_{[a,b]\times[c,d]} f(x)g(y)dA = \int_a^b f(x)dx \int_c^d g(y)dy$$

Example

We can use Fubini's theorem to compute volumes of solids. If $f(x,y) \geq 0$ on D, we can interpret the double integral

$$\iint_D f(x,y)\ dA$$

as the volume V of the solid that lies above D and under the surface $z = f(x,y)$. For example the volume of the solid that lies below the elliptic paraboloid

$$z = 16 - x^2 - 2y^2 = f(x,y)$$

and above the square $D = [0,2] \times [0,2]$ is

$$V = \iint_{[0,2]\times[0,2]} 16 - x^2 - 2y^2 dA \overset{Fubini}{=} \int_0^2 \left(\int_0^2 16 - x^2 - 2y^2\ dx \right) dy$$

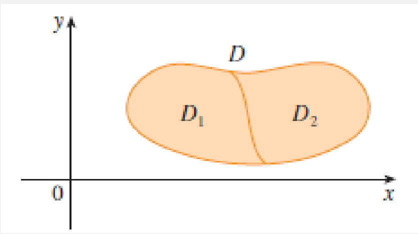
$$= \int_0^2 \left(16x - \frac{x^3}{3} - 2xy^2 \right) \Big|_0^2 dy = \int_0^2 \frac{88}{3} - 4y^2 dy = \left(\frac{88}{3} y - 4\frac{y^3}{3} \right) \Big|_0^2$$

$= 48$ units of volume

Additivity of domains property

Let D_1, D_2 be non-overlapping subsets of D that share no interior points with each other and $D = D_1 \cup D_2$. If f is continuous over D then f is continuous over D_1, D_2 and moreover

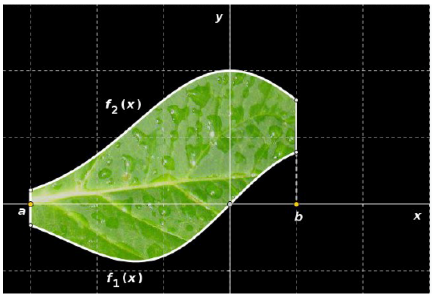
$$\iint_{D_1 \cup D_2} f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$

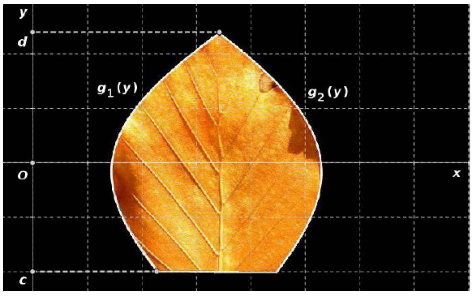


• compare with the Riemann integral of a real-valued function $f : [a,b] \to \mathbb{R}$ defined on $[a,b] = [a,c] \cup [c,b]$

Fact no 2: *In reality, rectangles will appear only in your sweet dreams*

• any decent real-life situation involves integration over irregular domains, which have the boundaries defined by different curves
• we'll interpret the boundaries as a union of multiple graphs of functions, and reduce everything to studying two important patterns



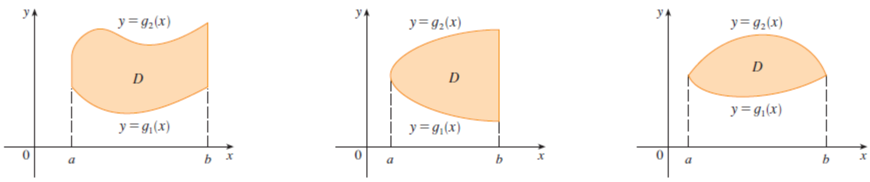


Normal domains

• the domain defined by .

$$D = \{(x, y) \in \mathbb{R}^2 : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

is called normal with respect to the x-axis (type I region)



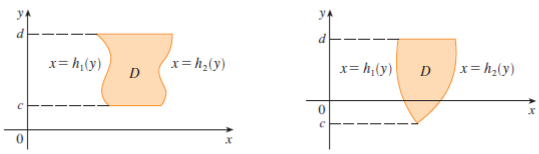
• if f is continuous on D, then

$$\iint_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$$

• the domain defined by

$$D = \{(x, y) \in \mathbb{R}^2 : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$$

is called normal with respect to the y-axis (type II region)



• if f is continuous on D, then

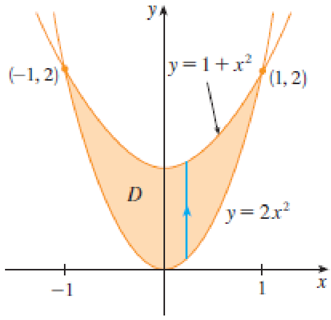
$$\iint_D f(x, y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y)dxdy$$

Example

We evaluate the following integral

$$\iint_D (x + 2y)\ dA.$$

where D is the plane region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



The points of intersection between the parabolas are the points $A(-1, 2)$ and $B(1, 2)$. The region bounded by these two parabolas is a normal domain with respect to the x-axis

$$D = \{(x, y) : -1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}$$

As a consequence, using a special form of Fubini's theorem one gets

$$\begin{aligned}
\iint_D (x + 2y)dA &= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\ dydx \\
&= \int_{-1}^{1} (xy + y^2)\Big|_{2x^2}^{1+x^2}\ dx \\
&= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\ dx \\
&= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\ dx \\
&= \left(-3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\right)\Bigg|_{-1}^{1} = \frac{32}{15}
\end{aligned}$$

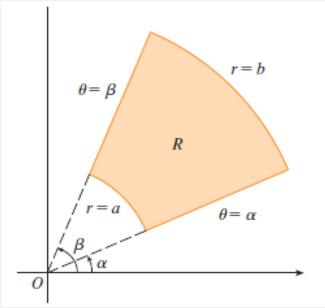
Remark

The area of a region $D \subset \mathbb{R}^2$ will be computed using the formula

$$area(D) = \iint_D 1\ dA$$

Double integrals in polar coordinates

If f is continuous over a polar rectangle



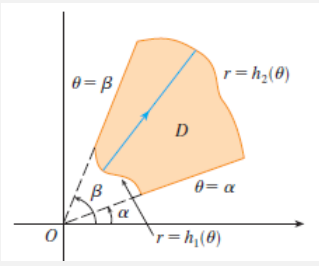
$$D = \{(r,\theta) : 0 \le a \le r \le b, \quad \alpha \le \theta \le \beta\}$$

where $\beta - \alpha \le 2\pi$, then

$$\iint_D f(x,y)dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta) r d\theta dr$$

If f is continuous over a polar region

$$D = \{(r,\theta) : 0 \le \alpha \le \theta \le \beta, \quad h_1(\theta) \le r \le h_2(\theta)\}$$



then one has

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta$$

Remark

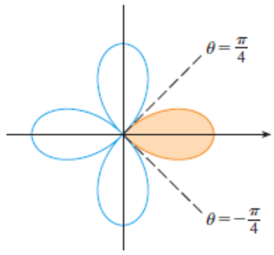
In integration theory expressions like x^2+y^2 usually need polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, with $r \ge 0$ and $\theta \in [0, 2\pi]$. Conversely one has

$$r = x^2 + y^2, \quad \theta = \arccos\frac{x}{\sqrt{x^2+y^2}} = \arcsin\frac{y}{\sqrt{x^2+y^2}}.$$

Example

We'll compute the area described by one of the leaves of the four-leaved rose, with boundary given by the curve

$$(x^2+y^2)^{\frac{3}{2}} = x^2 - y^2$$



In polar coordinates the equation of this curve becomes $r = \cos 2\theta$. Since $x = r\cos\theta$ and $y = r\sin\theta$ it leads to $x^2 - y^2 = r^2\cos 2\theta$ and $(x^2+y^2)^{\frac{3}{2}} = r^3$. The leaf written in polar coordinates is the polar region

$$D = \left\{(r,\theta) : \ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, \quad 0 \le r \le \cos 2\theta\right\}$$

$$\begin{aligned} area(D) &= \iint_D 1\ dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r dr d\theta \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{r^2}{2}\bigg|_0^{\cos 2\theta} d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2(2\theta)}{2} d\theta \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos 4\theta}{4} d\theta = \frac{1}{4}\left(\theta + \frac{\sin 4\theta}{4}\right)\bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8} \end{aligned}$$

Integration by substitution

Let us suppose f is continuous over a region $D \subset \mathbb{R}^2$ which is the image under a transform $x = x(u,v), y = y(u,v)$ of a region B, in the uv-plane. Then the following substitution rule holds

$$\iint_D f(x,y)\ dA = \iint_B f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv,$$

where $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$.

• the idea is to use a change of variables in order to get integration over a simpler region B (with a simpler geometric boundary)

• the term $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ appears in the one dimensional case without the modulus, havig the form $x'(t)$, where t is the new variable

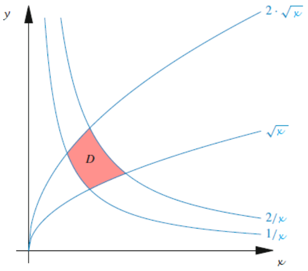
$$\int_{x(a)}^{x(b)} f(x)\ dx = \int_a^b f(x(t)) \cdot x'(t)\ dt$$

thus $B = [a,b]$ here

Example

Let us compute the integral

$$\iint_D \sqrt{xy}\; dA$$



over the domain given by

$$D = \left\{(x,y) \in \mathbb{R}^2 \;:\; x,y>0,\quad \sqrt{x} < y < 2\sqrt{x},\quad \frac{1}{x} < y < \frac{2}{x}\right\}$$

In order to find an appropriate substitution, let us have a look at the definition of D. One can write equivalently

$$1 < xy < 2 \quad \text{and} \quad 1 < \frac{y}{\sqrt{x}} < 2$$

Now, we'll choose new coordinates

$$u = xy \quad \text{und} \quad v = \frac{y}{\sqrt{x}} \qquad\qquad (*)$$

such that the new domain of integration becomes $B = [1,2] \times [1,2]$

In order to apply the substitution rule, one needs x and y viewed as functions in u, v

$$x = x(u,v) = u^{\frac{2}{3}} v^{-\frac{2}{3}} \quad \mathit{und} \quad y = y(u,v) = u^{\frac{1}{3}} v^{\frac{2}{3}},$$

since $(*) \implies u/v = x\sqrt{x} = \sqrt{x^3}$, thus $x = \left(\frac{u}{v}\right)^{\frac{2}{3}}$ and so forth.

By its very definition one gets

$$\frac{\partial(x,y)}{\partial(u,v)} = det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

but

$$\frac{\partial x}{\partial u} = \frac{2}{3} u^{-\frac{1}{3}} v^{-\frac{2}{3}}$$

$$\frac{\partial x}{\partial v} = -\frac{2}{3} u^{\frac{2}{3}} v^{-\frac{5}{3}}$$

$$\frac{\partial y}{\partial u} = \frac{1}{3} u^{-\frac{2}{3}} v^{\frac{2}{3}}$$

$$\frac{\partial y}{\partial v} = \frac{2}{3} u^{\frac{1}{3}} v^{-\frac{1}{3}}$$

hence

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{2}{3} u^{-\frac{1}{3}} v^{-\frac{2}{3}} \cdot \frac{2}{3} u^{\frac{1}{3}} v^{-\frac{1}{3}} + \frac{1}{3} u^{-\frac{2}{3}} v^{\frac{2}{3}} \cdot \frac{2}{3} u^{\frac{2}{3}} v^{-\frac{5}{3}} = \frac{4}{9} v^{-1} + \frac{2}{9} v^{-1} = \frac{2}{3v}$$

This determinant is positive. Now we can apply the substitution rule

$$\iint_D f(x,y)\,dA = \iint_B f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

is equivalent with

$$\begin{aligned}
\iint_D \sqrt{xy}\,dA &= \iint_{[1,2]\times[1,2]} \sqrt{(x(u,v)y(u,v)}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \\
&= \iint_{[1,2]\times[1,2]} \sqrt{u^{\frac{2}{3}}v^{-\frac{2}{3}}u^{\frac{1}{3}}v^{\frac{2}{3}}}\left|\frac{2}{3v}\right| dudv \\
&\overset{Fubini}{=} \int_1^2\left(\int_1^2 u^{\frac{1}{3}}v^{-\frac{1}{3}}u^{\frac{1}{3}}v^{\frac{2}{3}}\frac{2}{3v}du\right)dv \\
&= \frac{2}{3}\int_1^2\left(\int_1^2 u^{\frac{2}{3}}v^{-\frac{2}{3}}du\right)dv \\
&= \frac{2}{3}\int_1^2 v^{-\frac{2}{3}}\left(\int_1^2 u^{\frac{2}{3}}du\right)dv \\
&= \frac{2}{3}\frac{3}{5}(\sqrt[3]{32}-1)\int_1^2 v^{-\frac{2}{3}}dv = \frac{6}{5}(\sqrt[3]{32}-1)(\sqrt[3]{2}-1)
\end{aligned}$$

Applications of the double integral

Have a look in [1] or [2]

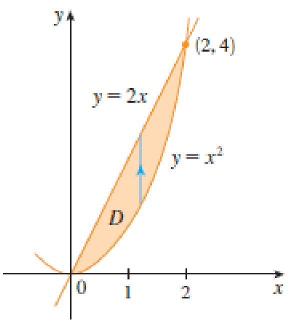
Solved problems

Problem 1

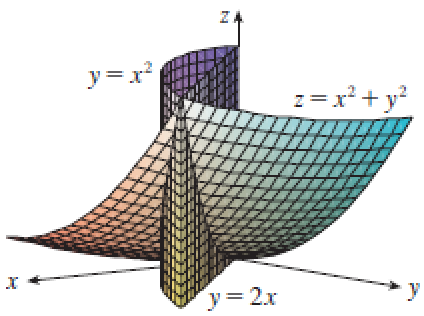
Let D be the domain bounded by the line $y = 2x$ and the parabola $y = x^2$. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D.

1st Solution: One can represent D as a normal domain with respect to the x-axis

$$D = \{(x,y) : 0 \leq x \leq 2,\ 2x \leq y \leq x^2\}$$



The illustration given below shows the solid bounded by the paraboloid $z = x^2 + y^2$ and the D region

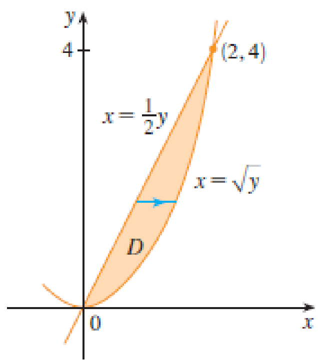


The volume of the solid will be

$$V = \iint_D (x^2 + y^2) dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) dy dx$$
$$= \int_0^2 \left(x^2 y + \frac{y^3}{3} \right) \Bigg|_{x^2}^{2x} dx = \int_0^2 \left(x^2(2x) - x^4 + \frac{(2x)^3}{3} - \frac{(x^2)^3}{3} \right) dx$$
$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx = \left(-\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \right) \Bigg|_0^2 = \frac{216}{35}.$$

2nd Solution: Simultaneoulsy one can represent D as a normal domain with respect to the y-axis

$$D = \{(x, y) : \frac{1}{2} y \le x \le \sqrt{y},\ 0 \le y \le 4\}$$



Thus, the volume can also be computed this way

$$V = \iint_D (x^2+y^2)dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2+y^2)dxdy$$

$$= \int_0^4 \left(\frac{x^3}{3} + y^2 x\right)\Bigg|_{\frac{y}{2}}^{\sqrt{y}} dy = \int_0^4 \left(\frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{y^3}{24} - \frac{y^3}{2}\right) dy$$

$$= \left(\frac{2}{15}y^{\frac{5}{2}} + \frac{2}{7}y^{\frac{7}{2}} - \frac{13}{96}y^4\right)\Bigg|_0^4 = \frac{216}{35}$$

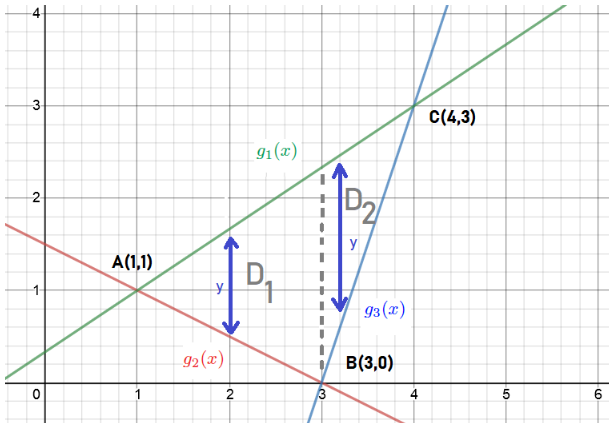
Problem 2

Find the integral of

$$f(x,y) = x + 3y$$

over the triangular region with vertices $A(1,1)$, $B(3,0)$, and $C(4,3)$.

Solution: We try to interpret ΔABC as a normal domain with respect to the y or x axes. It is not possible since only the northern boundary is given by a single curve. We'll have to split the set D into multiple subsets, each of which being a normal domain with respect to some axis. The dashed line marks the splitting. With the decomposition $D = D_1 \cup D_2$, becomes D_1 a normal domain with respect to the x-axis and D_2 with respect to x-axis also.



Thus, we start by writing

$$\int_D (x+3y)\ dA = \int_{D_1} (x+3y)\ dA + \int_{D_2} (x+3y)\ dA$$

After that, in order to compute the first integral, we need the function having the graph the line segment AC

$$AC: \quad \frac{x-1}{4-1} = \frac{y-1}{3-1}$$

$$AC: \quad y = \frac{2x}{3} + \frac{1}{3}$$

thus AC is the graph of $g_1(x) = \frac{2x}{3} + \frac{1}{3}$.

Further, one needs the function having AB as graph

$$AB: \quad \frac{x-1}{3-1} = \frac{y-1}{0-1}$$

$$AB: \quad y = -\frac{x}{2} + \frac{3}{2} = g_2(x)$$

Now, we can write

$$D_1 = \left\{ (x,y) \in \mathbb{R}^2 \;:\; 1 \leq x \leq 3, \quad -\frac{x}{2} + \frac{3}{2} \leq y \leq \frac{2x}{3} + \frac{1}{3} \right\}$$

and D_1 is a type I region

The first integral becomes

$$\iint_{D_1} (x+3y)\; dA = \int_1^3 \left(\int_{-\frac{x}{2}+\frac{3}{2}}^{\frac{2x}{3}+\frac{1}{3}} (x+3y)\; dy \right) dx$$

But

$$\int_{-\frac{x}{2}+\frac{3}{2}}^{\frac{2x}{3}+\frac{1}{3}} (x+3y)\; dy = xy + 3\frac{y^2}{2}\Bigg|_{-\frac{x}{2}+\frac{3}{2}}^{\frac{2x}{3}+\frac{1}{3}}$$

$$= x\left(\frac{2x}{3} + \frac{1}{3} - \left(-\frac{x}{2} + \frac{3}{2} \right) \right) + 3\left(\frac{\left(\frac{2x}{3}+\frac{1}{3}\right)^2}{2} - \frac{\left(-\frac{x}{2}+\frac{3}{2}\right)^2}{2} \right)$$

$$= \; etc. \; = \frac{2x^2 - 3x + 9}{72}$$

Hence

$$\iint_{D_1} (x+3y)\; dA = \int_1^3 \frac{2x^2-3x+9}{72}\; dx = \frac{1}{72}\left(2\frac{x^3}{3}\Bigg|_1^3 - 3x^2\Bigg|_1^3 + 9x\Bigg|_1^3 \right) = \ldots$$

For the second integral one needs to express mathematically the boundaries of D_2

$$BC: \quad \frac{x-3}{4-3} = \frac{y-0}{3-0}$$

$$BC: \quad y = 3x - 9 = g_3(x)$$

and D_2 is now defined by

$$D_2 = \left\{ (x,y) \in \mathbb{R}^2 \;:\; 3 \leq x \leq 4, \quad 3x - 9 \leq y \leq \frac{2x}{3} + \frac{1}{3} \right\}$$

Further

$$\iint_{D_2} (x+3y)\; dA = \int_3^4 \left(\int_{3x-9}^{\frac{2x}{3}+\frac{1}{3}} (x+3y)\; dy \right) dx$$

$$\int_{3x-9}^{\frac{2x}{3}+\frac{1}{3}} (x+3y)\ dy = xy + 3\frac{y^2}{2}\Bigg|_{3x-9}^{\frac{2x}{3}+\frac{1}{3}}$$

$$= x\left(\frac{2x}{3}+\frac{1}{3}-(3x-9)\right) + 3\left(\frac{\left(\frac{2x}{3}+\frac{1}{3}\right)^2}{2} - \frac{(3x-9)^2}{2}\right)$$

$$= \ etc. \ = \frac{x^2-8x+1}{72}$$

Hence

$$\iint_{D_2} (x+3y)\ dA = \int_3^4 \frac{x^2-8x+1}{72}\ dx = \frac{1}{72}\left(\frac{x^3}{3}\Bigg|_3^4 - 8x^2\Bigg|_3^4 + x\Bigg|_3^4\right) \ = \ etc.$$

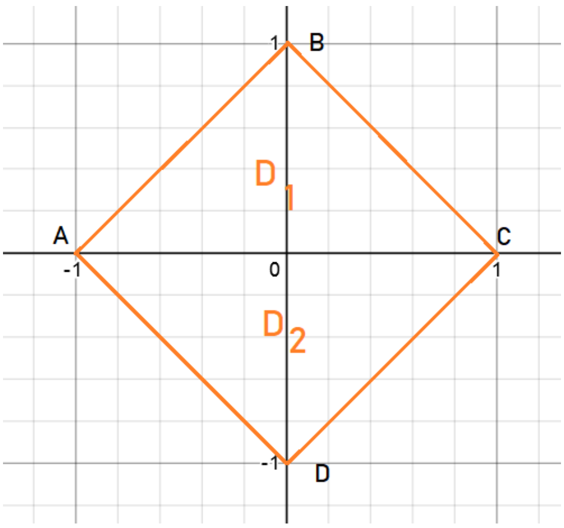
Problem 3

Evaluate the double integral

$$\iint\limits_D (x+y)dA,$$

where the domain of integration is the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$ from the xy-plane.

Solution: The given domain D is the square illustrated below



D is not a normal domain with respect to any of the axis (the up-down or left-right boundaries are not given by a singe curve). Decomposing D as

$$D = D_1 \cup D_2$$

both D_1 and D_2 will be normal domains. We'll evaluate only the first integral

$$\iint_D (x+y)dA = \iint_{D_1} (x+y)dA + \iint_{D_2} (x+y)dA$$

In order to evaluate D_1 we'll write D_1 as a normal domain. While y walks vertically between 0 and 1 the other coordinate x walks horizontally between the lines AB and BC. We need the equations of these two lines, from the point of view of x. (having an eye on the definiont of the type II region)

The line AB is given by $A(-1,0)$ and $B(0,1)$

$$AB: \quad \frac{x+1}{1} = \frac{y-0}{1-0}$$

$$AB: \quad x = y-1$$

The line BC is given by $B(0,1)$ and $C(1,0)$, thus

$$BC: \quad \frac{x-0}{1-0} = \frac{y-1}{0-1}$$

$$BC: \quad x = -y+1$$

Eventually

$$D_1 = \{(x,y): \ 0 \le y \le 1, \ y-1 \le x \le -y+1\}$$

and

$$\iint_{D_1} (x+y)dA = \int_0^1 \left(\int_{y-1}^{-y+1} (x+y) \ dx \right) dy$$

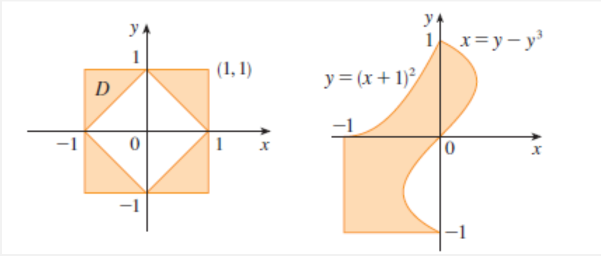
Starting from inside

$$\begin{aligned}\int_{y-1}^{-y+1} (x+y) \ dx &= \frac{x^2}{2}\Big|_{y-1}^{-y+1} + yx\Big|_{y-1}^{-y+1} \\ &= \frac{(-y+1)^2-(y-1)^2}{2} + y(-y+1-y+1) = -2y^2+2y\end{aligned}$$

$$\iint_{D_1} (x+y)dA = \int_0^1 (-2y^2+2y)dy = \ldots$$

Problem 4

Express D



as a union of normal domains and compute the integrals

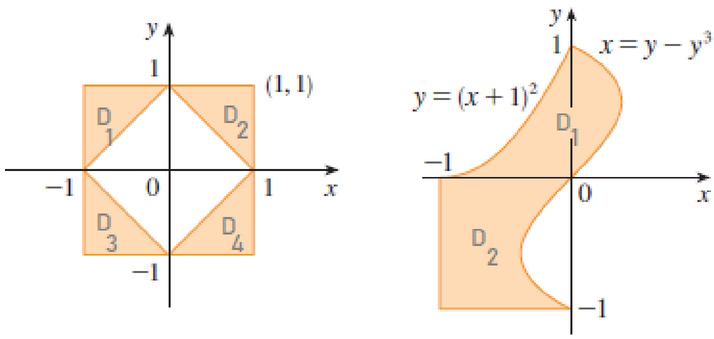
$$\iint_D x\ dA \quad \text{and} \quad \iint_D y\ dA$$

Solution: Let us observe, for the first domain, the decomposition

$$D = D_1 \cup D_2 \cup D_3 \cup D_4$$

and for the right one the decomposition

$$D = D_1 \cup D_2$$



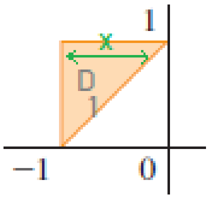
Thus, the first integral will be written as

$$\iint_D x\ dA = \iint_{D_1} x\ dA + \iint_{D_2} x\ dA + \iint_{D_3} x\ dA + \iint_{D_4} x\ dA$$

and all the four integrals are of the same type, thus we'll evaluate as an example only one of them. For all four triangles which describe D_1, D_2, D_3, D_4 it is useful to obtain their vertices. For example D_1 is bounded by the edges of a triangle generated by the vertices $(-1, 0)$, $(-1, 1)$ and $(0, 1)$.

$$\iint_{D_1} x\ dA = ???$$

As usual, the strategy is to write D_1 as a normal domain with respect to some axis. For example, vertically y can be bounded by 0 si 1 (see the drawing) and x will be horizontally enclosed between -1 on the left and the curve that joins the points $(-1, 0)$ si $(0, 1)$, on the right.



One needs the equation of the curve that bounds x on the right: the line segment between $(-1, 0)$ and $(0, 1)$

$$\frac{x+1}{1} = \frac{y-0}{1-0}$$

$$x + 1 = y$$

We have to interpret this equation from the point of view of x

$$x = y - 1$$

$$D_1 = \{(x, y) : \ 0 \le y \le 1, \ -1 \le x \le y - 1\} \ \text{ type II domain}$$

Consequently

$$\iint_{D_1} x \, dA = \int_0^1 \left(\int_{-1}^{y-1} x \, dx \right) dy =$$

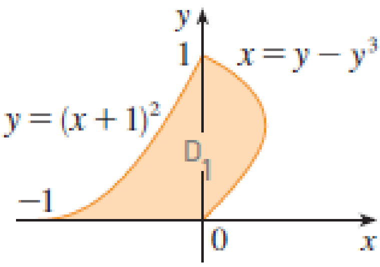
Hence

$$\int_{-1}^{y-1} x \, dx = \frac{x^2}{2}\Big|_{-1}^{y-1} = \frac{(y-1)^2 - 1}{2}$$

$$\iint_{D_1} x \, dA = \int_0^1 \left(\frac{(y-1)^2 - 1}{2} \right) dy = \frac{1}{2} \int_0^1 y^2 - 2y + 1 \, dy = \frac{1}{2} \left(\frac{1}{3} - 1 + 1 \right)$$

For the second domain we'll evaluate only the integral over D_1, the other one being simpler.

$$\iint_D y \, dA = \iint_{D_1} y \, dA + \iint_{D_2} y \, dA$$



Again y is vertically enclosed between the curves $y = 0$ (down) and $y = 1$ (up) and x between the curve $y = (x+1)^2$ (left) and $x = y - y^3$ (right). We have to express both curves from the point of view of x (compare with the definition of a type II region). For the second curve the job is done, since $x = y - y^3$. In this moment one can write

$$?? \leq x \leq y - y^3$$

In order to find the left term we express x as a function of y in $y = (x+1)^2$

$$y = (x+1)^2 \quad \Longrightarrow x^2 + 2x + 1 - y = 0$$

which is an equation in x, hence

$$x_{1,2} = \frac{-2 \pm \sqrt{4 - 4(1-y)}}{2} = -1 \pm \sqrt{y}$$

Only one of these two values correspond to our situation. Let us investigate a little bit the drawing. The x-coordinate of the points from the second quadrant will be between -1 and 0 and the y-coordinate between 0 and 1. So, for an x between -1 and 0 one has to choose $x = -1 + \sqrt{y}$, otherwise x can be smaller than -1.

$$D_1 = \{(x,y) : \; 0 \leq y \leq 1, \; -1 + \sqrt{y} \leq x \leq y - y^3\}$$

By its very definition

$$\iint_{D_1} y \; dA = \int_0^1 \left(\int_{-1+\sqrt{y}}^{y-y^3} y \; dx \right) dy$$

yadda, yadda, yadda.

Proposed problems

Problem 1. *For the domain* $D = \{(x, y) : \ 1 \le x \le 2, \ 1 \le y \le 4\}$ *compute the double integrals*

i) $\displaystyle\iint_D x^2 y^3 \ dxdy$

ii) $\displaystyle\iint_D \sqrt{x+y} \ dxdy$

iii) $\displaystyle\iint_D \ln(2x + 3y) \ dxdy$

Problem 2. *Find the volume of the solid bounded by the plane* $z = 0$ *and the paraboloid* $z = 1 - x^2 - y^2$.

Hint: use polar coordinates

Problem 3. *Evaluate the integral by changing to polar coordinates*

$$\iint_D \sin(x^2 + y^2) \ dA$$

where D *is the region in the first quadrant between the circles with center the origin and radii* $r = 1$ *and* $r = 3$.

Problem 4. *Evaluate the double integrals*

i) $\displaystyle\iint\limits_D x \cos y \ dA$, *where* D *is bounded by* $y = 0, y = x^2, x = 1$

ii) $\displaystyle\iint\limits_D (x^2 + 2y) \ dA$, *where* D *is bounded by* $y = x, y = x^3$ *and* $x \ge 0$

iii) $\displaystyle\iint\limits_D y^2 \ dA$, *where* D *is the triangular region with vertices* $(0, 1)$, $(1, 2)$ *and* $(4, 1)$

iv) $\displaystyle\iint\limits_D xy) \ dA$, *where* D *is enclosed by* $y = \sqrt{1 - x^2}$, $x \ge 0$ *and the axes. (quarter circle)*

Bibliography

[1] J. Stewart. Calculus, *Thompson Brooks/Cole*, 2008.

[2] O. Lipovan. Analiza matematica: Calcul Integral, *Ed. Politehnica*, 2006.