ÜBUNGSBLATT 6

Aufgabe 1. *Sei* $D = \{(u, v) : 0 \le u \le 2\pi, 0 \le v \le \pi\}$ *und:*

$$r(u,v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}$$

eine Parameterdarstellung der Einheitssphäre. Für das Vektorfeld:

$$F(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

berechnen Sie das Flächenintegral $\iint\limits_S F \cdot dS$

Solutie: Integrala de suprafata vectoriala este definita prin

$$\iint\limits_S \boldsymbol{F} \cdot dS = \iint\limits_D \boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot (\boldsymbol{r}_u \times \boldsymbol{r}_v)\, dA$$

unde produsul "$\cdot$" inseamna produsul scalar a doi vectori 3-dimensionali.

Vom calcula pe rand produsul vectorial $(\boldsymbol{r}_u \times \boldsymbol{r}_v)$ din formula de mai sus apoi produsul scalar cerut

Prin definitie

$$\begin{aligned}(\boldsymbol{r}_u \times \boldsymbol{r}_v) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\sin u \sin v & \cos u \sin v & 0 \\ \cos u \cos v & \sin u \cos v & -\sin v \end{vmatrix} \\ &= -\cos u \sin^2 v \cdot \boldsymbol{i} - \sin u \sin^2 v \cdot \boldsymbol{j} + (-\sin^2 u \sin v \cos v - \cos^2 u \sin v \cos v) \cdot \boldsymbol{k} \\ &= -\cos u \sin^2 v \cdot \boldsymbol{i} - \sin u \sin^2 v \cdot \boldsymbol{j} - \sin v \cos v \cdot \boldsymbol{k}\end{aligned}$$

Tinand cont de legea de corespondenta data de F se poate observa usor ca

$$F(r(u,v)) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}$$

pe care la fel de bine putem sa-l gandim ca un vector linie. Prin urmare produsul scalar din definitia integralei vectoriale de suprafata este

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot (\boldsymbol{r}_u \times \boldsymbol{r}_v) &= \cos u \sin v(-\cos u \sin^2 v) + \sin u \sin v(-\sin u \sin^2 v) + \cos v(-\sin v \cos v) \\ &= -\sin^3 v(\cos^2 u + \sin^2 u) - \cos^2 v \sin v = -\sin^3 v - \cos^2 v \sin v \\ &= -\sin v(\sin^2 v + \cos^2 v) = -\sin v\end{aligned}$$

Acum putem aplica formula integralei vectoriale de suprafata

$$\iint\limits_S \boldsymbol{F} \cdot dS = \iint\limits_D \boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot (\boldsymbol{r}_u \times \boldsymbol{r}_v)\ dA = \iint\limits_D -\sin v\ dA$$

dar domeniul D inseamna $[0, 2\pi] \times [0, \pi]$ prin urmare

$$\iint\limits_S \boldsymbol{F} \cdot dS = \int_0^{2\pi} \int_0^{\pi} -\sin v\ dvdu$$

Incepem, ca de obicei la integrale duble, din interior pentru a evalua integrala

$$\int_0^{\pi} -\sin v\ dv = \cos v \Big|_0^{\pi} = -1 - 1 = -2$$

apoi

$$\iint\limits_S \boldsymbol{F} \cdot dS = \int_0^{2\pi} -2\ du = -4\pi$$

Aufgabe 2. *(Satz von Gauss am Zylinder)*
Berechnen Sie den Fluss des Vektorfeldes:

$$\boldsymbol{F} = (xz, yz, z^2)$$

durch die Oberfläche des Zylinderausschnitts:

$$Z = \{(x, y, z) : x^2 + y^2 \leq 1, 0 \leq z \leq 2\}$$

von innen nach aussen. Verwenden Sie den Satz von Gauss.

Solutie: Formula lui Gauss este

$$\iint\limits_{\partial Z} \boldsymbol{F} \cdot dS = \iiint\limits_Z \mathrm{div}\boldsymbol{F}\ dV$$

unde Z este cilindrul si ∂Z suprafata sa laterala. Avantajul acestei abordari consta in faptul ca putem evita integrarea pe suprafata laterala integrand in schimb pe un corp 3-dimensional Z (cilindrul). Pe cilindru avem nevoie de integrala tripla, dar fara a o introduce putem calcula integrala tripla necesara lucrand in acelasi mod in care lucram pentru o integrala dubla. Sa observam intai ca cilindrul se poate exprima ca

$$Z = \{(x, y, z) :\ 0 \leq x \leq 1,\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},\ 0 \leq z \leq 2\}$$

Calculam acum divergenta campului F

$$\mathrm{div}F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = z + z + 2z = 4z$$

$$\iint\limits_{\partial Z} \boldsymbol{F} \cdot dS = \iiint\limits_Z \mathrm{div}\boldsymbol{F}\ dV = \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^2 4z\ dzdydx$$

Luam pe rand integralele incepand din interior

$$\int_0^2 4z\ dz = 4z^2/2\Big|_0^2 = 2$$

apoi

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\ dy = 4\sqrt{1-x^2}$$

iar in final

$$\iint_{\partial Z} \boldsymbol{F} \cdot dS = \int_0^1 4\sqrt{1-x^2}\ dx = etc.$$

Am ajuns deja la o integrala obisnuita care se evalueaza cu schimbare de variabila, preferabil.(vezi cursul 1)

Aufgabe 3. *Berechnen Sie das Integral:*

$$\iint_S x^2 dS$$

wobei S die Sphäre:

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 16\}$$

ist.

Hinweis: Die Sphäre besitzt die Parameterdarstellung:

$$\boldsymbol{r}(u,v) = \begin{pmatrix} 4\cos u \sin v \\ 4\sin u \sin v \\ 4\cos v \end{pmatrix}$$

mit $(u,v) \in (0,2\pi) \times (0,\pi)$

Solutie: Avem de evaluat o integrala scalara de suprafata, tot pe o sfera integram. Formula integralei scalare difera putin de cea vectoriala

$$\iint_S f(x,y,z) dS = \iint_D f(\boldsymbol{r}(u,v)) \|\boldsymbol{r}_u \times \boldsymbol{r}_v\|\ dA$$

Pentru o sfera de raza 1 am calculat la prima problema produsul vectorial $\boldsymbol{r}_u \times \boldsymbol{r}_v$. Aici sfera are raza 4 si se poate observa ca vectorul de pozitie $r(u,v)$ este de patru ori mai mare decat cel de la prima problema. Din aceasta cauza

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = 16(-\cos u \sin^2 v \cdot \boldsymbol{i} - \sin u \sin^2 v \cdot \boldsymbol{j} - \sin v \cos v \cdot \boldsymbol{k})$$

deoarece va aparea cate un 4 pe ultimele doua linii ale determinantului care da produsul vectorial.

$$\begin{aligned}\|\boldsymbol{r}_u \times \boldsymbol{r}_v\| &= 16\sqrt{(-\cos u \sin^2 v)^2 + (-\sin u \sin^2 v)^2 + (-\sin v \cos v)^2}\\ &= 16\sqrt{\sin^4 v(\cos^2 u + \sin^2 u) + \sin^2 v \cos^2 v}\\ &= 16\sqrt{\sin^4 v + \sin^2 v \cos^2 v} = 16 \sin v\end{aligned}$$

Deoarece functia care trebuie integrata este $f(x, y, z) = x^2$ obtinem

$$f(r(u, v)) = 16 \cos^2 u \sin^2 v$$

apoi

$$\begin{aligned}\iint\limits_S f(x,y,z)dS &= \iint\limits_D f(\boldsymbol{r}(u,v))\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|\ dA = \iint\limits_D 16\cos^2 u \sin^2 v 16 \sin v\ dA\\ &= 16^2 \int_0^{2\pi}\int_0^{\pi} \cos^2 u \sin^3 v\ dvdu = etc\end{aligned}$$

Aufgabe 4. *Berechne den Inhalt der Mantelfläche des Zylinders:*

$$Z = \{(R, u, v) : 0 \le R \le 2, \quad 0 \le u \le 2\pi, -1 \le v \le 1\}$$

Hinweis: eine Parametrisierung ist:

$$\boldsymbol{r}(u, v) = \begin{pmatrix} 2\cos u \\ 2\sin u \\ v \end{pmatrix}$$

Solutie: Trebuie sa calculam aria suprafetei laterale a unui cilindru.

Parametrizarea cilindrului fiind data, singura necunoscuta este legatura dintre ariile suprafetelor si integralele de suprafata. Pentru o suprafata S aria sa este data prin formula

$$aria(S) = \iint_S 1\ dS$$

adica integrala scalara de suprafata a functiei constante 1.

In cazul nostru aria suprafetei cerute este

$$aria(Z) = \iint_Z 1\ dS = \iint_D 1 \cdot \|\boldsymbol{r}_u \times \boldsymbol{r}_v\|\ dA = \int_0^{2\pi}\int_{-1}^{1} \|\boldsymbol{r}_u \times \boldsymbol{r}_v\|\ dvdu$$

unitatea de masura fiind unitati de arie. Calculam $\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|$ pentru parametrizarea cilindrului

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\cos u \cdot \boldsymbol{i} + 2\sin u \cdot \boldsymbol{j}$$

Apoi
$$\|\boldsymbol{r}_u \times \boldsymbol{r}_v\| = \sqrt{(2\cos u)^2 + (2\sin u)^2} = 4$$
In final aria ceruta va fi
$$aria(Z) = \iint_Z 1 \ dS = \int_0^{2\pi} \int_{-1}^{1} 4 \ dvdu = 16\pi$$