

“In mathematics you don’t understand things. You just get used to them.”

John von Neumann

9

The Double Integral

General facts

- for a complete introduction of this topic have a look in book of Stewart [\[1\]](#) (I sent you the chapter about the double integral a few weeks ago)
- in the sequel we’ll discuss double integration from a computational point of view, no special attention will be given to the existence conditions or the domain of integration

Fact no 1: Integration over rectangles is easy

Theorem of Fubini:

Let f be a **continuous** function over the rectangle

$$R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\},$$

then the following identities hold

$$\iint_{[a,b] \times [c,d]} f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$



Consequences

For a function with separated variables $f(x, y) = f(x)g(y)$ one gets

$$\iint_{[a,b] \times [c,d]} f(x)g(y) dA = \int_a^b f(x) dx \int_c^d g(y) dy$$



Example

We can use Fubini's theorem to compute volumes of solids. If $f(x, y) \geq 0$ on D , we can interpret the double integral

$$\iint_D f(x, y) dA$$

as the volume V of the solid that lies above D and under the surface $z = f(x, y)$. For example the volume of the solid that lies below the elliptic paraboloid

$$z = 16 - x^2 - 2y^2 = f(x, y)$$

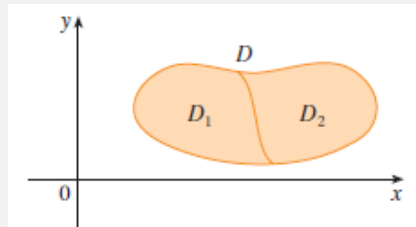
and above the square $D = [0, 2] \times [0, 2]$ is

$$\begin{aligned}
V &= \iint_{[0,2] \times [0,2]} 16 - x^2 - 2y^2 dA \stackrel{\text{Fubini}}{=} \int_0^2 \left(\int_0^2 16 - x^2 - 2y^2 dx \right) dy \\
&= \int_0^2 \left(16x - \frac{x^3}{3} - 2xy^2 \right) \Big|_0^2 dy = \int_0^2 \frac{88}{3} - 4y^2 dy = \left(\frac{88}{3}y - 4\frac{y^3}{3} \right) \Big|_0^2 \\
&= 48 \text{ units of volume}
\end{aligned}$$

Additivity of domains property

Let D_1, D_2 be non-overlapping subsets of D that share no interior points with each other and $D = D_1 \cup D_2$. If f is continuous over D then f is continuous over D_1, D_2 and moreover

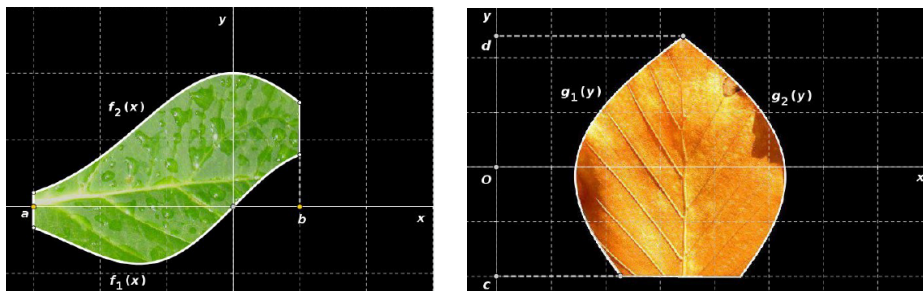
$$\iint_{D_1 \cup D_2} f(x, y) dA = \iint_{D_1} f(x, y) dA + \iint_{D_2} f(x, y) dA$$



• compare with the Riemann integral of a real-valued function $f : [a, b] \rightarrow \mathbb{R}$ defined on $[a, b] = [a, c] \cup [c, b]$

Fact no 2: In reality, rectangles will appear only in your sweet dreams

- any decent real-life situation involves integration over irregular domains, which have the boundaries defined by different curves
- we'll interpret the boundaries as a union of multiple graphs of functions, and reduce everything to studying two important patterns

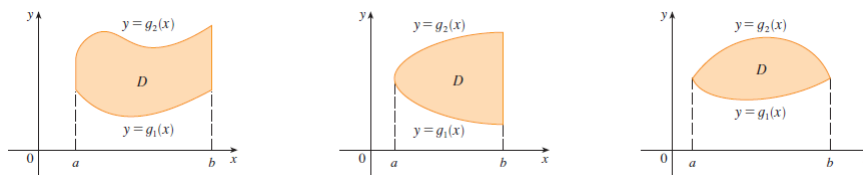


Normal domains

The domain defined by

$$D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

is called **normal with respect to the x-axis** (type I region)



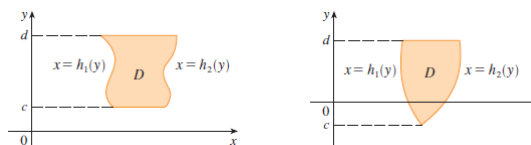
If f is continuous on D , then

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

The domain defined by

$$D = \{(x, y) \in \mathbb{R}^2 : h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$$

is called **normal with respect to the y-axis** (type II region)



If f is continuous on D , then

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

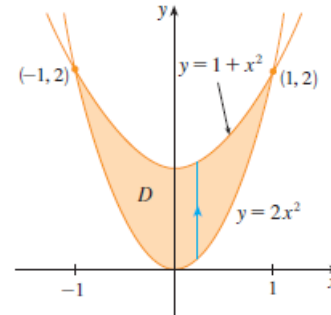


Example

We evaluate the following integral

$$\iint_D (x + 2y) \, dA.$$

where D is the plane region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



The points of intersection between the parabolas are the points $A(-1, 2)$ and $B(1, 2)$. The region bounded by these two parabolas is a normal domain with respect to the x -axis

$$D = \{(x, y) : -1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}$$

As a consequence, using a special form of Fubini's theorem one gets

$$\begin{aligned} \iint_D (x + 2y) \, dA &= \int_{-1}^1 \int_{2x^2}^{1+x^2} (x + 2y) \, dy \, dx \\ &= \int_{-1}^1 (xy + y^2) \Big|_{2x^2}^{1+x^2} \, dx \\ &= \int_{-1}^1 [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2] \, dx \\ &= \int_{-1}^1 (-3x^4 - x^3 + 2x^2 + x + 1) \, dx \\ &= \left(-3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \right) \Big|_{-1}^1 = \frac{32}{15} \end{aligned}$$



Remarks

The area of a region $D \subset \mathbb{R}^2$ will be computed using the formula

$$\text{area}(D) = \iint_D 1 \, dA$$

Double integrals in polar coordinates

If f is continuous over a polar rectangle

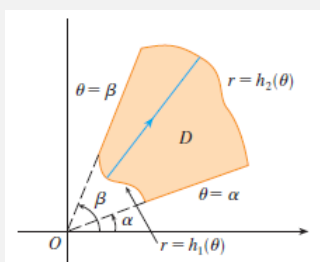
$$D = \{(r, \theta) : 0 \leq a \leq r \leq b, \quad \alpha \leq \theta \leq \beta\}$$

where $\beta - \alpha \leq 2\pi$, then

$$\iint_D f(x, y) dA = \int_a^b \int_\alpha^\beta f(r \cos \theta, r \sin \theta) r d\theta dr$$

If f is continuous over a polar region

$$D = \{(r, \theta) : 0 \leq \alpha \leq \theta \leq \beta, \quad h_1(\theta) \leq r \leq h_2(\theta)\}$$



then one has

$$\iint_D f(x, y) dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$



Remarks

In integration theory expressions like $x^2 + y^2$ usually need polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$, with $r \geq 0$ and $\theta \in [0, 2\pi]$. Conversely one has

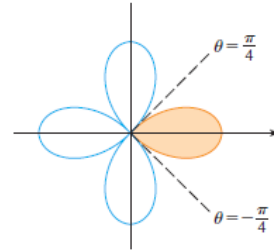
$$r = \sqrt{x^2 + y^2}, \quad \theta = \arccos \frac{x}{\sqrt{x^2 + y^2}} = \arcsin \frac{y}{\sqrt{x^2 + y^2}}.$$



Example

We'll compute the area described by one of the leaves of the four-leaved rose, with boundary given by the curve

$$(x^2 + y^2)^{\frac{3}{2}} = x^2 - y^2$$



In polar coordinates the equation of this curve becomes $r = \cos 2\theta$. Since $x = r \cos \theta$ and $y = r \sin \theta$ it leads to $x^2 - y^2 = r^2 \cos 2\theta$ and $(x^2 + y^2)^{\frac{3}{2}} = r^3$. The leaf written in polar coordinates is the polar region

$$D = \left\{ (r, \theta) : -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}, 0 \leq r \leq \cos 2\theta \right\}$$

$$\begin{aligned} \text{area}(D) &= \iint_D 1 \, dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r \, dr \, d\theta \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{r^2}{2} \Big|_0^{\cos 2\theta} \, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2(2\theta)}{2} \, d\theta \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1 + \cos 4\theta}{4} \, d\theta = \frac{1}{4} \left(\theta + \frac{\sin 4\theta}{4} \right) \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8} \end{aligned}$$

Integration by substitution

Let us suppose f is continuous over a region $D \subset \mathbb{R}^2$ which is the image under a transform $x = x(u, v), y = y(u, v)$ of a region B , in the uv -plane. Then the following substitution rule holds

$$\iint_D f(x, y) \, dA = \iint_B f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv,$$

where $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$.

- the idea is to use a change of variables in order to get integration over a simpler region B (with a simpler geometric boundary)
- the term $\left| \frac{\partial(x, y)}{\partial(u, v)} \right|$ appears in the one dimensional case without the modulus, havig the form $u'(t)$, where t is the new variable

$$\int_{u(a)}^{u(b)} f(x) \, dx = \int_a^b f(u(t)) \cdot u'(t) \, dt$$

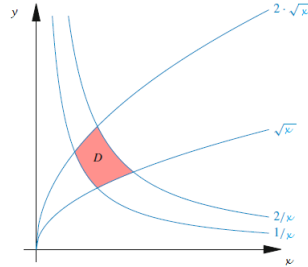
thus $B = [a, b]$ here

 **Example**

Let us compute the integral

$$\iint_D \sqrt{xy} \, dA$$

over the domain given by



$$D = \left\{ (x, y) \in \mathbb{R}^2 : x, y > 0, \sqrt{x} < y < 2\sqrt{x}, \frac{1}{x} < y < \frac{2}{x} \right\}$$

In order to find an appropriate substitution, let us have a look at the definition of D. One can write equivalently

$$1 < xy < 2 \quad \text{and} \quad 1 < \frac{y}{\sqrt{x}} < 2$$

Now, we'll choose new coordinates

$$u = xy \quad \text{und} \quad v = \frac{y}{\sqrt{x}} \quad (*)$$

such that the new domain of integration becomes $B = [1, 2] \times [1, 2]$

In order to apply the substitution rule, one needs x and y viewed as functions in u, v

$$x = x(u, v) = u^{\frac{2}{3}} v^{-\frac{2}{3}} \quad \text{und} \quad y = y(u, v) = u^{\frac{1}{3}} v^{\frac{2}{3}},$$

since $(*) \implies u/v = x\sqrt{x} = \sqrt{x^3}$, thus $x = \left(\frac{u}{v}\right)^{\frac{2}{3}}$ and so forth.

By its very definition one gets

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

but

$$\frac{\partial x}{\partial u} = \frac{2}{3} u^{-\frac{1}{3}} v^{-\frac{2}{3}}$$

$$\frac{\partial x}{\partial v} = -\frac{2}{3} u^{\frac{2}{3}} v^{-\frac{5}{3}}$$

$$\frac{\partial y}{\partial u} = \frac{1}{3} u^{-\frac{2}{3}} v^{\frac{2}{3}}$$

$$\frac{\partial y}{\partial v} = \frac{2}{3} u^{\frac{1}{3}} v^{-\frac{1}{3}}$$

hence

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{2}{3} u^{-\frac{1}{3}} v^{-\frac{2}{3}} \cdot \frac{2}{3} u^{\frac{1}{3}} v^{-\frac{1}{3}} + \frac{1}{3} u^{-\frac{2}{3}} v^{\frac{2}{3}} \cdot \frac{2}{3} u^{\frac{2}{3}} v^{-\frac{5}{3}} = \frac{4}{9} v^{-1} + \frac{2}{9} v^{-1} = \frac{2}{3v}$$

This determinant is positive. Now we can apply the substitution rule

$$\iint_D f(x, y) \, dA = \iint_B f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dudv$$

is equivalent with

$$\begin{aligned} \iint_D \sqrt{xy} \, dA &= \iint_{[1,2] \times [1,2]} \sqrt{(x(u, v))y(u, v)} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dudv \\ &= \iint_{[1,2] \times [1,2]} \sqrt{u^{\frac{2}{3}}v^{-\frac{2}{3}}u^{\frac{1}{3}}v^{\frac{2}{3}}} \left| \frac{2}{3v} \right| \, dudv \\ &\stackrel{Fubini}{=} \int_1^2 \left(\int_1^2 u^{\frac{1}{3}}v^{-\frac{1}{3}}u^{\frac{1}{3}}v^{\frac{2}{3}} \frac{2}{3v} \, du \right) \, dv \\ &= \frac{2}{3} \int_1^2 \left(\int_1^2 u^{\frac{2}{3}}v^{-\frac{2}{3}} \, du \right) \, dv \\ &= \frac{2}{3} \int_1^2 v^{-\frac{2}{3}} \left(\int_1^2 u^{\frac{2}{3}} \, du \right) \, dv \\ &= \frac{2}{3} \frac{3}{5} (\sqrt[3]{32} - 1) \int_1^2 v^{-\frac{2}{3}} \, dv = \frac{6}{5} (\sqrt[3]{32} - 1) (\sqrt[3]{2} - 1) \end{aligned}$$



Applications of the double integral

Have a look in [\[1\]](#) or [\[2\]](#)

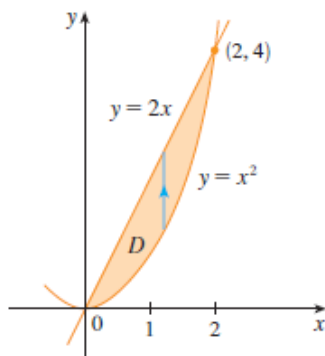


Solved problems

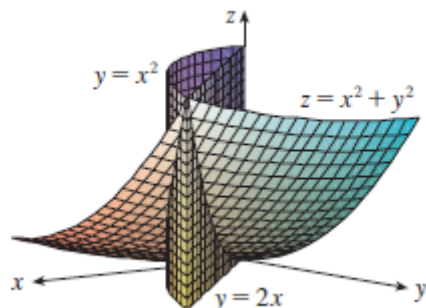
Problem 1. Let D be the domain bounded by the line $y = 2x$ and the parabola $y = x^2$. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D .

1st Solution: One can represent D as a normal domain with respect to the x -axis

$$D = \{(x, y) : 0 \leq x \leq 2, 2x \leq y \leq x^2\}$$



The illustration given below shows the solid bounded by the paraboloid $z = x^2 + y^2$ and the D region

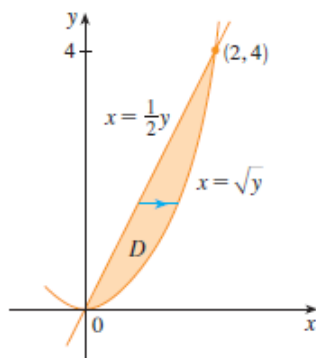


The volume of the solid will be

$$\begin{aligned}
 V &= \iint_D (x^2 + y^2) dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) dy dx \\
 &= \int_0^2 \left(x^2 y + \frac{y^3}{3} \right) \Big|_{x^2}^{2x} dx = \int_0^2 \left(x^2(2x) - x^4 + \frac{(2x)^3}{3} - \frac{(x^2)^3}{3} \right) dx \\
 &= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx = \left(-\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \right) \Big|_0^2 = \frac{216}{35}.
 \end{aligned}$$

2nd Solution: Simultaneously one can represent D as a normal domain with respect to the y -axis

$$D = \{(x, y) : \frac{1}{2}y \leq x \leq \sqrt{y}, 0 \leq y \leq 4\}$$



Thus, the volume can also be computed this way

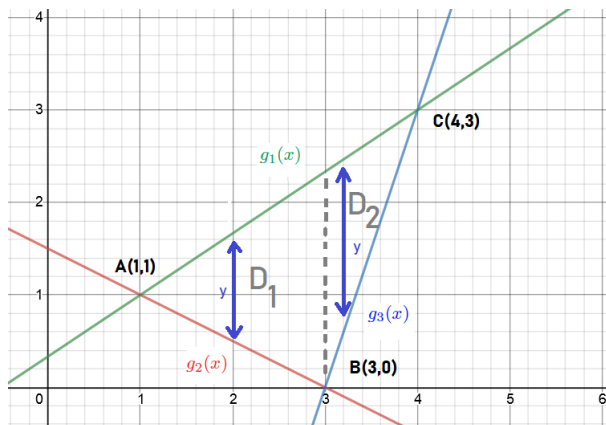
$$\begin{aligned}
 V &= \iint_D (x^2 + y^2) dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2) dx dy \\
 &= \int_0^4 \left(\frac{x^3}{3} + y^2 x \right) \Big|_{\frac{1}{2}y}^{\sqrt{y}} dy = \int_0^4 \left(\frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{y^3}{24} - \frac{y^3}{2} \right) dy \\
 &= \left(\frac{2}{15} y^{\frac{5}{2}} + \frac{2}{7} y^{\frac{7}{2}} - \frac{13}{96} y^4 \right) \Big|_0^4 = \frac{216}{35}
 \end{aligned}$$

Problem 2. Find the integral of

$$f(x, y) = x + 3y$$

over the triangular region with vertices $A(1, 1)$, $B(3, 0)$, and $C(4, 3)$.

Solution: We try to interpret $\triangle ABC$ as a normal domain with respect to the y or x axes. It is not possible since only the northern boundary is given by a single curve. We'll have to split the set D into multiple subsets, each of which being a normal domain with respect to some axis. The dashed line marks the splitting. With the decomposition $D = D_1 \cup D_2$, becomes D_1 a normal domain with respect to the x -axis and D_2 with respect to x -axis also.



Thus, we start by writing

$$\int_D (x + 3y) dA = \int_{D_1} (x + 3y) dA + \int_{D_2} (x + 3y) dA$$

After that, in order to compute the first integral, we need the function having the graph the line segment AC

$$\begin{aligned}
 AC : \quad \frac{x-1}{4-1} &= \frac{y-1}{3-1} \\
 AC : \quad y &= \frac{2x}{3} + \frac{1}{3}
 \end{aligned}$$

thus AC is the graph of $g_1(x) = \frac{2x}{3} + \frac{1}{3}$.

Further, one needs the function having AB as graph

$$AB : \frac{x-1}{3-1} = \frac{y-1}{0-1}$$

$$AB : y = -\frac{x}{2} + \frac{3}{2} = g_2(x)$$

Now, we can write

$$D_1 = \left\{ (x, y) \in \mathbb{R}^2 : 1 \leq x \leq 3, -\frac{x}{2} + \frac{3}{2} \leq y \leq \frac{2x}{3} + \frac{1}{3} \right\}$$

and D_1 is a type I region

The first integral becomes

$$\iint_{D_1} (x+3y) \, dA = \int_1^3 \left(\int_{-\frac{x}{2} + \frac{3}{2}}^{\frac{2x}{3} + \frac{1}{3}} (x+3y) \, dy \right) dx$$

But

$$\begin{aligned} \int_{-\frac{x}{2} + \frac{3}{2}}^{\frac{2x}{3} + \frac{1}{3}} (x+3y) \, dy &= xy + 3\frac{y^2}{2} \Big|_{-\frac{x}{2} + \frac{3}{2}}^{\frac{2x}{3} + \frac{1}{3}} \\ &= x \left(\frac{2x}{3} + \frac{1}{3} - \left(-\frac{x}{2} + \frac{3}{2} \right) \right) + 3 \left(\frac{\left(\frac{2x}{3} + \frac{1}{3} \right)^2}{2} - \frac{\left(-\frac{x}{2} + \frac{3}{2} \right)^2}{2} \right) \\ &= \dots\dots\dots = \frac{2x^2 - 3x + 9}{72} \end{aligned}$$

Hence

$$\iint_{D_1} (x+3y) \, dA = \int_1^3 \frac{2x^2 - 3x + 9}{72} \, dx = \frac{1}{72} \left(2\frac{x^3}{3} \Big|_1^3 - 3x^2 \Big|_1^3 + 9x \Big|_1^3 \right) = \dots$$

For the second integral one needs to express mathematically the boundaries of D_2

$$BC : \frac{x-3}{4-3} = \frac{y-0}{3-0}$$

$$BC : y = 3x - 9 = g_3(x)$$

and D_2 is now defined by

$$D_2 = \left\{ (x, y) \in \mathbb{R}^2 : 3 \leq x \leq 4, 3x - 9 \leq y \leq \frac{2x}{3} + \frac{1}{3} \right\}$$

Further

$$\iint_{D_2} (x+3y) \, dA = \int_3^4 \left(\int_{3x-9}^{\frac{2x}{3} + \frac{1}{3}} (x+3y) \, dy \right) dx$$

$$\int_{3x-9}^{\frac{2x}{3}+\frac{1}{3}} (x+3y) dy = xy + 3\frac{y^2}{2} \Big|_{3x-9}^{\frac{2x}{3}+\frac{1}{3}}$$

$$= x\left(\frac{2x}{3} + \frac{1}{3} - (3x-9)\right) + 3\left(\frac{\left(\frac{2x}{3} + \frac{1}{3}\right)^2}{2} - \frac{(3x-9)^2}{2}\right)$$

$$= \dots\dots\dots = \frac{x^2 - 8x + 1}{72}$$

Hence

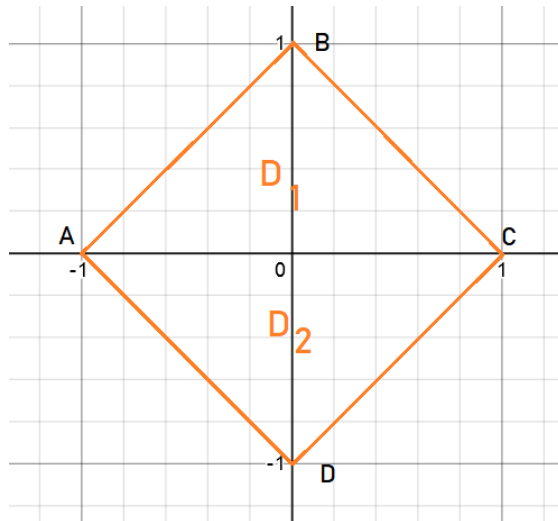
$$\iint_{D_2} (x+3y) dA = \int_3^4 \frac{x^2 - 8x + 1}{72} dx = \frac{1}{72} \left(\frac{x^3}{3} \Big|_3^4 - 8x^2 \Big|_3^4 + x \Big|_3^4 \right) = \dots$$

Problem 3. Evaluate the double integral

$$\iint_D (x+y) dA,$$

where the domain of integration is the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ from the xy -plane.

Solution: The given domain D is the square illustrated below



D is not a normal domain with respect to any of the axis (the up-down or left-right boundaries are not given by a single curve). Decomposing D as

$$D = D_1 \cup D_2$$

both D_1 and D_2 will be normal domains. We'll evaluate only the first integral

$$\iint_D (x+y)dA = \iint_{D_1} (x+y)dA + \iint_{D_2} (x+y)dA$$

In order to evaluate D_1 we'll write D_1 as a normal domain. While y walks vertically between 0 and 1 the other coordinate x walks horizontally between the lines AB and BC . We need the equations of these two lines, from the point of view of x . (having an eye on the definition of the type II region)

The line AB is given by $A(-1,0)$ and $B(0,1)$

$$AB : \frac{x+1}{1} = \frac{y-0}{1-0}$$

$$AB : x = y - 1$$

The line BC is given by $B(0,1)$ and $C(1,0)$, thus

$$BC : \frac{x-0}{1-0} = \frac{y-1}{0-1}$$

$$BC : x = -y + 1$$

Eventually

$$D_1 = \{(x, y) : 0 \leq y \leq 1, y - 1 \leq x \leq -y + 1\}$$

and

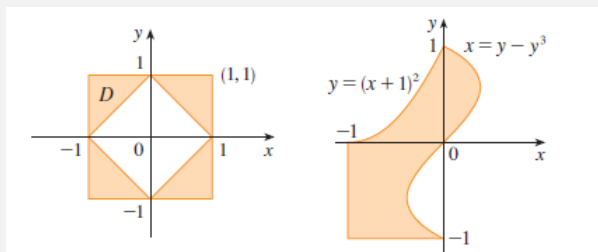
$$\iint_{D_1} (x+y)dA = \int_0^1 \left(\int_{y-1}^{-y+1} (x+y) dx \right) dy$$

Starting from inside

$$\begin{aligned} \int_{y-1}^{-y+1} (x+y) dx &= \frac{x^2}{2} \Big|_{y-1}^{-y+1} + yx \Big|_{y-1}^{-y+1} \\ &= \frac{(-y+1)^2 - (y-1)^2}{2} + y(-y+1 - y+1) = -2y^2 + 2y \end{aligned}$$

$$\iint_{D_1} (x+y)dA = \int_0^1 (-2y^2 + 2y) dy = \dots$$

Problem 4. Express D



as a union of normal domains and compute the integrals

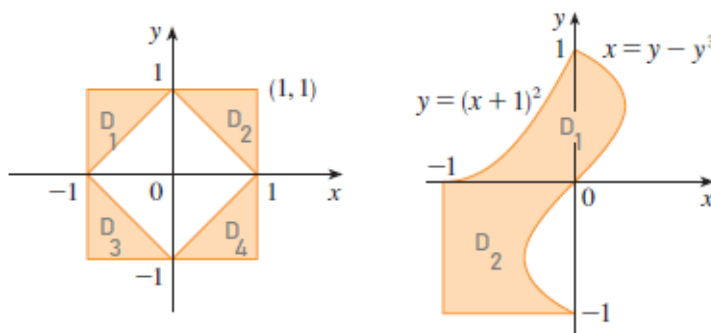
$$\iint_D x \, dA \quad \text{and} \quad \iint_D y \, dA$$

Solution: Let us observe, for the first domain, the decomposition

$$D = D_1 \cup D_2 \cup D_3 \cup D_4$$

and for the right one the decomposition

$$D = D_1 \cup D_2$$



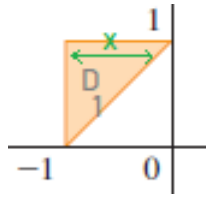
Thus, the first integral will be written as

$$\iint_D x \, dA = \iint_{D_1} x \, dA + \iint_{D_2} x \, dA + \iint_{D_3} x \, dA + \iint_{D_4} x \, dA$$

and all the four integrals are of the same type, thus we'll evaluate as an example only one of them. For all four triangles which describe D_1, D_2, D_3, D_4 it is useful to obtain their vertices. For example D_1 is bounded by the edges of a triangle generated by the vertices $(-1, 0)$, $(-1, 1)$ and $(0, 1)$.

$$\iint_{D_1} x \, dA = ???$$

As usual, the strategy is to write D_1 as a normal domain with respect to some axis. For example, vertically y can be bounded by 0 si 1 (see the drawing) and x will be horizontally enclosed between -1 on the left and the curve that joins the points $(-1, 0)$ si $(0, 1)$, on the right.



One needs the equation of the curve that bounds x on the right: the line segment between $(-1, 0)$ and $(0, 1)$

$$\frac{x+1}{1} = \frac{y-0}{1-0}$$

$$x+1 = y$$

We have to interpret this equation from the point of view of x

$$x = y - 1$$

$$D_1 = \{(x, y) : 0 \leq y \leq 1, -1 \leq x \leq y - 1\} \text{ type II domain}$$

Consequently

$$\iint_{D_1} x \, dA = \int_0^1 \left(\int_{-1}^{y-1} x \, dx \right) dy =$$

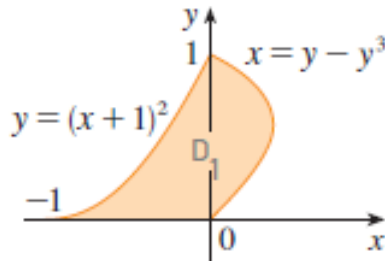
Hence

$$\int_{-1}^{y-1} x \, dx = \frac{x^2}{2} \Big|_{-1}^{y-1} = \frac{(y-1)^2 - 1}{2}$$

$$\iint_{D_1} x \, dA = \int_0^1 \left(\frac{(y-1)^2 - 1}{2} \right) dy = \frac{1}{2} \int_0^1 (y^2 - 2y + 1) dy = \frac{1}{2} \left(\frac{1}{3} - 1 + 1 \right)$$

For the second domain we'll evaluate only the integral over D_1 , the other one being simpler.

$$\iint_D y \, dA = \iint_{D_1} y \, dA + \iint_{D_2} y \, dA$$



Again y is vertically enclosed between the curves $y = 0$ (down) and $y = 1$ (up) and x between the curve $y = (x+1)^2$ (left) and $x = y - y^3$ (right). We have to express both curves from the point of view of x (compare with the definition

of a type II region). For the second curve the job is done, since $x = y - y^3$. In this moment one can write

$$?? \leq x \leq y - y^3$$

In order to find the left term we express x as a function of y in $y = (x + 1)^2$

$$y = (x + 1)^2 \implies x^2 + 2x + 1 - y = 0$$

which is an equation in x , hence

$$x_{1,2} = \frac{-2 \pm \sqrt{4 - 4(1 - y)}}{2} = -1 \pm \sqrt{y}$$

Only one of these two values correspond to our situation. Let us investigate a little bit the drawing. The x -coordinate of the points from the second quadrant will be between -1 and 0 and the y -coordinate between 0 and 1 . So, for an x between -1 and 0 one has to choose $x = -1 + \sqrt{y}$, otherwise x can be smaller than -1 .

$$D_1 = \{(x, y) : 0 \leq y \leq 1, -1 + \sqrt{y} \leq x \leq y - y^3\}$$

By its very definition

$$\iint_{D_1} y \, dA = \int_0^1 \left(\int_{-1 + \sqrt{y}}^{y - y^3} y \, dx \right) dy$$

yadda, yadda, yadda.

Proposed problems

Problem 1. For the domain $D = \{(x, y) : 1 \leq x \leq 2, 1 \leq y \leq 4\}$ compute the double integrals

i) $\iint_D x^2 y^3 \, dx dy$

ii) $\iint_D \sqrt{x+y} \, dx dy$

iii) $\iint_D \ln(2x+3y) \, dx dy$

Problem 2. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Hint: use polar coordinates

Problem 3. Evaluate the integral by changing to polar coordinates

$$\iint_D \sin(x^2 + y^2) \, dA$$

where D is the region in the first quadrant between the circles with center the origin and radii $r = 1$ and $r = 3$.

Problem 4. Evaluate the double integrals

i) $\iint_D x \cos y \, dA$, where D is bounded by $y = 0, y = x^2, x = 1$

ii) $\iint_D (x^2 + 2y) \, dA$, where D is bounded by $y = x, y = x^3$ and $x \geq 0$

iii) $\iint_D y^2 \, dA$, where D is the triangular region with vertices $(0, 1), (1, 2)$ and $(4, 1)$

iv) $\iint_D xy \, dA$, where D is enclosed by $y = \sqrt{1-x^2}, x \geq 0$ and the axes.
(quarter circle)

Bibliography

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- [2] O. Lipovan. *Analiza matematica: Calcul Integral*, *Ed. Poli-tehnica*, 2006.