Übungsblatt 5

Aufgabe 1. *Gegeben seien die Funktion* $f : \mathbb{R}^3 \to \mathbb{R}$ *und die Kurve* $c : [1, 3] \to \mathbb{R}^3$ *mit:*

$$f(x, y, z) = \frac{2y}{x}, \qquad c(t) = \left(2t, t^2, \frac{t^3}{3}\right)$$

Berechnen Sie die Länge $L(c)$ *und das Kurvenintegral* $\int_c f(x, y, z)ds$ *für diese Kurve.*

Lösung: Formula pentru calculul integralei curbilinii scalare e

$$\int_c f(s)ds = \int_a^b f(\boldsymbol{r}(t)) \cdot \|\boldsymbol{r}'(t)\| \; dt$$

unde $c : [a, b] \to \mathbb{R}^3$ este curba a carui vector de pozitie intr-un punct oarecare este $\boldsymbol{r}(t)$. Pentru a aplica formula trebuie sa lucram in etape si sa calculam pe rand toti termenii de care avem nevoie.

Pentru inceput sa observam ca vectorul de pozitie este

$$\boldsymbol{r}(t) = \left(2t, t^2, \frac{t^3}{3}\right)$$

Derivata sa relativ la t devine

$$\boldsymbol{r}'(t) = \left(2, 2t, t^2\right)$$

Si in acest moment putem obtine o parte din integrala din dreapta

$$\|\boldsymbol{r}'(t)\| = \sqrt{2^2 + (2t)^2 + (t^2)^2} = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2 + 2)^2} = t^2 + 2$$

Pentru a evalua $f(r(t))$ trebuie sa tine cont de legea de asociere data de f

$$(x, y, z) \mapsto \frac{2y}{x}$$

si atunci deoarece $r(t) = \left(2t, t^2, \frac{t^3}{3}\right)$, daca respectam aceeasi regula

$$\left(2t, t^2, \frac{t^3}{3}\right) \mapsto \frac{2t^2}{2t} = t$$

Asadar $f(r(t)) = t$ si intrucat curba pe care dorim sa integram este definita pe intervalul $[1, 3]$ avem, in final

$$\int_c f(s)ds = \int_1^3 t \cdot (t^2 + 2) \; dt = \int_1^3 t^3 + 2t \; dt = \left.\frac{t^4}{4}\right|_1^3 + 2\left.\frac{t^2}{2}\right|_1^3 = 28$$

Lungimea unui segment de curba se calculeaza cu formula

$$L(c) = \int_a^b \|\boldsymbol{r}'(t)\| dt$$

deci

$$L(c) = \int_1^3 \|\boldsymbol{r}'(t)\| dt = \int_1^3 t^2 + 2 \; dt = \frac{t^3}{3}\Big|_1^3 + 2t\Big|_1^3 = \frac{26}{3} + 5$$

Aufgabe 2. *Berechnen Sie die Längen folgender Kurven:*

i) $c : [0, \frac{\pi}{2}] \to \mathbb{R}^2, \quad c(t) = (2R\cos^3 t, 2R\sin^3 t), \quad R > 0,$

ii) $c : [0, \infty) \to \mathbb{R}^3, \quad c(t) = e^{-2t}(\cos t, \sin t, 1).$

Lösung: i) Formula lungimii unei curbe intre punctul $c(a)$ si $c(b)$ este data de

$$L(c) = \int_a^b \|\boldsymbol{r}'(t)\| dt$$

Dificultatea este de natura tehnica deoarece vor aparea expresii trigonometrice mai dificil de evaluat.

Intai vectorul de pozitie al unui punct oarecare de pe curba este

$$r(t) = (2R\cos^3 t, 2R\sin^3 t)$$

Derivata sa

$$r'(t) = (-2R \cdot 3\cos^2 t \sin t, 2R \cdot 3\sin^2 t \cos t)$$

Acum putem evalua norma (lungimea) sa

$$\|r'(t)\| = \sqrt{(-2R \cdot 3\cos^2 t \sin t)^2 + (2R \cdot 3\sin^2 t \cos t)^2} = \sqrt{36R^2\cos^4 t \sin^2 t + 36R^2 \sin^4 t \cos^2 t}$$

E momentul sa cautam un factor comun, pentru a putea avansa

$$\|r'(t)\| = \sqrt{36R^2\cos^2 t \sin^2 t \cdot (\cos^2 t + \sin^2 t)} = \sqrt{36R^2\cos^2 t\sin^2 t \cdot 1} = 6R\cos t \sin t$$

In ultima egalitate se afla o mica capcana, folosim faptul ca $\sin t$ si $\cos t$ sunt pozitive pe intervalul $[0, \frac{\pi}{2}]$ unde t este cuprins. In alte conditii am avea $\sqrt{\cos^2 t} = \pm\cos t$, etc.

Pe cai, ca se filmeaza !

$$L(c) = \int_a^b \|\boldsymbol{r}'(t)\| dt = \int_0^{\frac{\pi}{2}} 6R\cos t \sin t \; dt = 6R \int_0^{\frac{\pi}{2}} \cos t \cdot \sin t \; dt$$

Daca evaluam integrala prin formula integrarii prin parti posibil sa ne invartim in cerc deoarece derivand una dintre functiile trig. $\sin, \cos$ se obtine cealalta. Trebuie sa incercam o alta abordare.

Ce altceva ar functiona aici ? (Gandire laterala) Cum altfel am putea interpreta produsul $\cos t \cdot \sin t$?

Perspectiva 1: Produsul $\cos t \cdot \sin t$ apare in formula $\sin(2t) = 2\sin t \cos t$ si atunci integrala se transforma in

$$L(c) = 3R\int_0^{\frac{\pi}{2}} \sin(2t)\ dt$$

care cere o schimbare de variabila $2t = y$, etc.

Perspectiva 2: Produsul $\cos t \cdot \sin t$ poate avea legatura cu prima metoda de schimbare a variabilei. Cu o schimbare de variabila $\cos t = y$, diferentiind obtinem $-\sin t\ dt = dy$ si astfel putem face rost de toate elementele necesare realizarii schimbarii de variabila

$$\begin{aligned} L(c) &= 6R\int_0^{\frac{\pi}{2}} \cos t \cdot \sin t\ dt = 6R\int_{\cos 0}^{\cos\frac{\pi}{2}} y(-dy) = -6R\int_1^0 y\ dy \\ &= 6R\int_0^1 y\ dy = 6R\frac{y^2}{2}\bigg|_0^1 = 3R \end{aligned}$$

Aufgabe 3. *Werten Sie das Integral* $\int_c f(s)ds$ *aus, wobei* $f : \mathbb{R}^3 \to \mathbb{R}$ *und* $c : [0,1] \to \mathbb{R}^3$ *definiert durch:*

$$f(x,y,z) = \frac{x^2}{9} + \frac{y^2}{9} + \frac{3z^2}{5}, \quad c(t) = (3\cos t, 3\sin t, 4t)$$

sind.

Lösung: Se rezolva in mod asemanator cu Aufgabe 1. De observat ca legea de asociere data de f presupune urmatoarele transformari

$$(x, y, z) \mapsto \frac{x^2}{9} + \frac{y^2}{9} + \frac{3z^2}{5}$$

Deci $f(r(t))$ se obtine prin aceeasi regula

$$(3\cos t, 3\sin t, 4t) \mapsto \frac{(3\cos t)^2}{9} + \frac{(3\sin t)^2}{9} + \frac{3(4t)^2}{5}$$

$$f(r(t)) = \frac{(3\cos t)^2}{9} + \frac{(3\sin t)^2}{9} + \frac{3(4t)^2}{5} = \frac{9(\cos^2 t + \sin^2 t)}{9} + \frac{48t^2}{5} = 1 + \frac{48t^2}{5}$$

Have fun in continuare...

Aufgabe 4. *Überprüfen Sie, ob die folgenden Vektorfelder Potentiale haben (auf $\mathbb{R}^2$ oder $\mathbb{R}^3$). Berechnen Sie gegebenfalls:*

i) $\boldsymbol{F}(x,y) = e^x(\sin x, \cos x)$
ii) $\boldsymbol{F}(x,y) = (xy^4 + 2x^5, 2x^2y^3 - y^6)$
iii) $\boldsymbol{F}(x,y) = (-x, y)$
iv) $\boldsymbol{F}(x,y,z) = (6x^2, 2x^3 - z, y)$
v) $\boldsymbol{F}(x,y,z) = (y, 0, 0)$

Lösung: Va trebui sa testam daca functiile vectoriale date admit potential si in caz afirmativ sa calculam respectivul potential.

Folosim Potentialkriterium (vezi curs) care are o forma particulara pentru campuri vectoriale cu valori in $\mathbb{R}^3$ sau $\mathbb{R}^2$

In $\mathbb{R}^3$ acesta se reduce la faptul ca rotorul campului trebuie sa fie nul

$$\mathrm{rot}F = 0 \qquad \Longrightarrow \qquad \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = 0$$

unde f_1, f_2, f_3 sunt componentele campului vectorial dat prin $F(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$

In $\mathbb{R}^2$ un camp $F : \mathbb{R}^2 \to \mathbb{R}^2$ are potential daca este indeplinita conditia

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$$

unde $F(x,y) = (f_1(x,y), f_2(x,y))$

Sa luam spre exemplu subpunctul iv) unde $\boldsymbol{F}(x,y,z) = (6x^2, 2x^3 - z, y)$. Extragem intai componentele

$$f_1(x,y,z) = 6x^2 \qquad f_2(x,y,z) = 2x^3 - z \qquad f_3(x,y,z) = y$$

Determinantul formal care da expresia rotorului trebuie manevrat cu grija, ideea e sa il dezvoltam dupa prima linie, altfel nu prea au sens unele expresii care pot aparea

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \boldsymbol{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \boldsymbol{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \boldsymbol{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} = 1 - (-1) = 2$$

$$\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} = 0 - 0 = 0$$

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 6x^2 - 0 = 6x^2$$

Putem acum observa ca $\mathrm{rot}F = 2\boldsymbol{i} + 0\boldsymbol{j} + 6x^2\boldsymbol{k}$ care nu este nul. Deci criteriul nu este indeplinit si F nu admite potential

La subpunctul iii) avem $\boldsymbol{F}(x,y)=(-x,y)$ de unde extragem componentele

$$f_2(x,y)=-x \qquad f_2(x,y)=y$$

Conditia de integrabilitate este

$$\frac{\partial f_1}{\partial y}=\frac{\partial f_2}{\partial x}$$

si intradevar

$$\frac{\partial f_1}{\partial y}=0=\frac{\partial f_2}{\partial x}$$

deci campul vectorial admite potential care se va calcula cu formula (vezi in curs)

$$f(x,y)=\int_a^x f_1(t,b)dt+\int_b^y f_2(x,s)ds.$$

unde a,b sunt constante oarecare, drept care le putem alege ca fiind 0.

$$f(x,y)=\int_0^x f_1(t,0)dt+\int_0^y f_2(x,s)ds.$$

Remarcam ca $f_1(t,0)=-t$ iar $f_2(x,s)=s$ daca respectam legile de asociere date de f_1,f_2, apoi expresia potentialului va fi

$$f(x,y)=\int_0^x t\ dt+\int_0^y s\ ds=\left.\frac{t^2}{2}\right|_0^x+\left.\frac{s^2}{2}\right|_0^y=\frac{x^2}{2}+\frac{y^2}{2}$$

Aufgabe 5. *Gegeben seien das Vektorfeld $\boldsymbol{F}:\mathbb{R}^3\to\mathbb{R}^3$ und die Kurve $\gamma:[0,\pi]\to\mathbb{R}^3$ mit:*

$$\boldsymbol{F}(x,y,z)=(z+y,x+z,y+x),\quad \textit{und}\quad \gamma(t)=(1-\sin t,\cos t,t).$$

Berechnen Sie die Rotation des Vektorfelds $\boldsymbol{F}$ und das Kurvenintegral

$$\int_\gamma \boldsymbol{F}\cdot d\boldsymbol{r}.$$

a) *Zeigen Sie, dass das Vektorfeld $\boldsymbol{F}$ Potentialfunktionen besitzt.*
b) *Berechnen Sie eine Potentialfunktion $\Phi:\mathbb{R}^3\to\mathbb{R}$ zu $\boldsymbol{F}$.*
c) *Berechnen Sie das Kurvenintegral $\int_\gamma \boldsymbol{F}\cdot d\boldsymbol{r}$ mit Hilfe der Potentialfunktion Φ.*

Lösung: Prin definitie integrala curbilinie a unui camp vectorial F este

$$\int_\gamma \boldsymbol{F}\cdot d\boldsymbol{r}=\int_a^b \boldsymbol{F}(\boldsymbol{r}(t))\cdot \boldsymbol{r}'(t)\ dt$$

aici $\cdot$ inseamna produsul scalar dintre vectori si nu un simplu produs intre numere reale. Prin urmare in cazul unui camp vectorial 3-dimensional formula de calcul al integralei curbilinii vectoriale

$$\int_\gamma \boldsymbol{F}\cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(x(t),y(t),z(t))\cdot(x'(t),y'(t),z'(t))dt$$
$$= \int_a^b f_1(x(t),y(t),z(t))x'(t)\,dt + \int_a^b f_2(x(t),y(t),z(t))y'(t)\,dt$$
$$+ \int_a^b f_3(x(t),y(t),z(t))z'(t)\,dt$$

unde $F=(f_1,f_2,f_3)$ si vectorul de pozitie al unui punct oarecare de pe curba este vazut ca fiind $r(t)=(x(t),y(t),z(t))$.

Sa revenim la datele problemei si sa observam ca

$$f_1(x,y,z)=z+y \qquad f_2(x,y,z)=x+z \qquad f_3(x,y,z)=y+x$$

iar $r(t)=(1-\sin t,\cos t,t)=(x(t),y(t),z(t))$ deci $r'(t)=(-\cos t,-\sin t,1)$

Asadar

$$x'(t)=-\cos t \qquad y'(t)=-\sin t \qquad z'(t)=1$$

si

$$f_1(x(t),y(t),z(t))=t+\cos t \qquad f_2(x(t),y(t),z(t))=1-\sin t+t$$
$$f_3(x(t),y(t),z(t))=\cos t+1-\sin t$$

$$\int_\gamma \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^\pi (t+\cos t)(-\cos t)+(1-\sin t+t)(-\sin t)+(\cos t+1-\sin t)1\,dt$$

Dupa cum observam pentru a evalua integralele rezultate trebuie sa utilizam de foarte multe ori integrarea prin parti. Pentru unele dintre ele e nevoie de formule trigonometrice

$$\sin^2 t=\frac{1-\cos(2t)}{2}$$
$$\cos^2 t=\frac{\cos(2t)-1}{2}$$

$$\int_0^\pi \sin t\cdot\sin t\,dt=\int_0^\pi \frac{1-\cos(2t)}{2}\,dt=\frac{1}{2}t\Big|_0^\pi-\frac{\sin(2t)}{4}\Big|_0^\pi=\frac{\pi}{2}$$

Rezultatul final va fi 0 dupa cum se va arata la subpunctul c) !!!

a) Pentru a arata ca F are o functie potential trebuie sa aratam ca $rot\ F$ este vectorul nul.

$$rot\ F=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}=\boldsymbol{i}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)-\boldsymbol{j}\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right)+\boldsymbol{k}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

$$\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}=1-1=0$$

$$\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} = 1 - 1 = 0$$
$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 1 - 1 = 0$$

deci $rot\ F = 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$ deci F indeplineste Potentialkriterium.

b) Pentru a calcula functia potential facem din nou apel la formula

$$f(x,y,z) = \int_a^x f_1(t,b,c)dt + \int_b^y f_2(x,s,c)ds + \int_c^z f_3(x,y,u)du.$$

unde putem alege orice constanta pentru a, b, c. Alegem 0 si de aceasta data

$$\begin{aligned} f(x,y,z) &= \int_0^x f_1(t,0,0)dt + \int_0^y f_2(x,s,0)ds + \int_0^z f_3(x,y,u)du \\ &= \int_0^x (0+0)dt + \int_0^y (x+0)ds + \int_0^z (x+y)du \\ &= 0 + xs\Big|_0^y + (x+y)u\Big|_0^z = xy + xz + yz \end{aligned}$$

In final

$$f(x,y,z) = xy + yz + zx$$

este functia potential a lui F. Avantajul obtinerii acestei functii se va vedea la subpunctul urmator

c) Avand o functie potential putem calcula mai rapid integrala curbilinie vectoriala cu ajutorul acesteia, caci functia potential se comporta ca un fel de primitiva a lui F

$$\int_\gamma F \cdot dr = f(\gamma(0)) - f(\gamma(\pi))$$

Asadar functia potential se calculeaza in capetele curbei. Se obtine

$$\gamma(0) = (1 - \sin 0, \cos 0, 0) = (1, 1, 0)$$
$$\gamma(\pi) = (1 - \sin\pi, \cos\pi, \pi) = (1, -1, \pi)$$
$$\int_\gamma F \cdot dr = f((1,1,0)) - f((1,-1,\pi)) = 1 + (-1) - \pi + \pi = 0$$