"The $50-50-90$ rule: Anytime you have a $50-50$ chance of getting something right, there's a 90% probability you'll get it wrong."

Andy Rooney

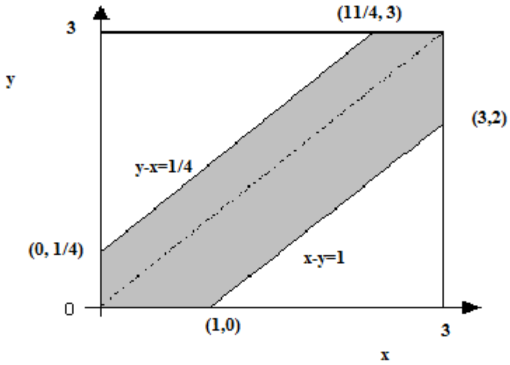
5 Classical problems in probability theory

Chance of meeting in a restaurant

A man and a woman decide to meet in a restaurant after 21 o'clock. The restaurant closes at 24 o'clock. Because of their busy schedule they decide that whoever arrives first at the restaurant will wait, for a while, for the other one. The man would be ready to wait an hour and the woman only 15 minutes!

What's the probability that they will meet?

Solution: We will model mathematically the problem in the following way: let us denote by x the time when the women arrives at the restaurant and by y the time when the man arrives. We can consider 21 o'clock to be 0 and then 24 will mean 3. Thus $x, y \in [0, 3]$. All the possibilities are represented by the points (x, y) located inside the square $[0, 3] \times [0, 3]$ drawn below.



If the man arrives first, that means $y \leq x$, then they will meet if $x - y \leq 1$ (the time when the woman arrives is at most one hour later). The arrival times which satisfy this restriction are contained in the gray region, between the first bisectrix $y = x$ and the line $x - y = 1$

If the woman arrives first, i.e. $x \leq y$, then they will meet only if $y - x \leq \frac{1}{4}$. All the possible arrival times which satisfy this restriction are contained in the gray region, between the first bisectrix $y = x$ and the line $y - x = \frac{1}{4}$

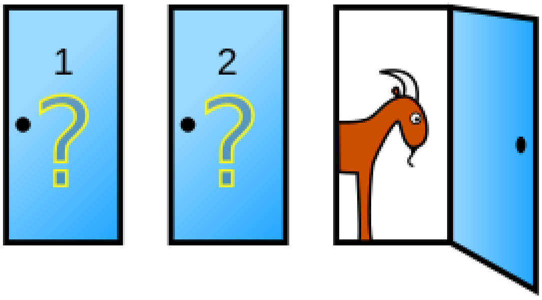
The chance that they will meet is computed using the formula:

$$P = \frac{\text{number of favourable cases}}{\text{number of possible cases}}.$$

Of course, there is an infinity of favourable and possible cases. However we can estimate the probability without counting all the points (x, y) but using the areas of the regions corresponding to the favourable cases and to all the possible cases. The probability they do meet is:

$$P = \frac{\text{area of the gray region}}{\text{area of the square}} = \frac{\frac{103}{32}}{3^2} \approx 35\%.$$

The Monty Hall Problem



Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car, behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat.

He then says to you, "Do you want to pick door No. 2?"

Is it to your advantage to switch your choice ?

Counting techniques:

• multiplication rule of counting: if a task consists of a sequence of n choices in which there are p_1 ways to make the first choice, p_2 ways to make the second, etc., then the task can be done in:

$$p_1 \cdot p_2 \cdot \ldots \cdot p_n$$

different ways.

• permutations of n distinct objects taken k at a time: the number of arrangements of k objects chosen from n objects in which:

· the n objects are distinct

· repeats are not allowed

· order matters

is given by the formula $A_n^k = \dfrac{n!}{(n-k)!}$

• combination of n distinct objects taken k at a time: the number of arrangements of n objects using $k \leq n$ of them, in which:

· the n objects are distinct

· repeats are not allowed

· order does not matter

is given by the formula $C_n^k = \frac{n!}{(n-k)!k!}$

• arrangements with non-distinct items (repetitions): the number of permutations of n objects, where there are n_1 of the 1st type, n_2 of the 2nd type,..., n_k of k-th type is:

$$\frac{n!}{n_1!n_2!\cdot\ldots\cdot n_k!}, \qquad n_1+n_2+\ldots+n_k=n.$$

Theorem: (Inclusion-Exclusion Principle)
For any finite sequence $A_1, A_2, \ldots, A_n$ of subsets of a finite set X we have:

$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{i=1}^{n}|A_i| - \sum_{1\le i<j\le n}|A_i\cap A_j| + \ldots + (-1)^{p-1}\sum_{1\le i_1<i_2<\ldots<i_p\le n}|A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_p}|$$

$$+\ldots+(-1)^{n-1}|A_1\cap A_2\cap\ldots A_n|.$$

The principle expressed in its complementary form:

$$\left|\bigcap_{k=1}^{n}\bar{A}_k\right| = \left|X\setminus\bigcup_{k=1}^{n}A_k\right| = |X| - \sum_{i=1}^{n}|A_i| + \sum_{1\le i<j\le n}|A_i\cap A_j| + \ldots$$

$$+(-1)^p\sum_{1\le i_1<i_2<\ldots<i_p\le n}|A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_p}| + \ldots + (-1)^n|A_1\cap A_2\cap\ldots A_n|.$$

Classical schemes of probability theory::

• Poincaré's theorem:

$$P\left(\bigcup_{k=1}^{n}E_k\right) = \sum_{k=1}^{n}P(E_k) - \sum_{k=1}^{n-1}\sum_{j=k+1}^{n}P(E_k\cap E_j) +$$

$$+\sum_{k=1}^{n-2}\sum_{j=k+1}^{n-1}\sum_{i=j+1}^{n}P(E_k\cap E_j\cap E_i) - \ldots + (-1)^{n-1}P(E_1\cap E_2\ldots\cap E_n)$$

e.g. for $n=3$:

$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)$$

$$+P(A\cap B\cap C)$$

• the multiplication rule:

$$P\left(\bigcap_{k=1}^{n}E_k\right) = P(E_1)\cdot P(E_2|E_1)\cdot P(E_3|E_1\cap E_2)\cdot\ldots\cdot P\left(E_n|\bigcap_{k=1}^{n-1}E_k\right)$$

if the events are independent:

$$P\left(\bigcap_{k=1}^{n}E_k\right) = P(E_1)\cdot P(E_2)\cdot\ldots\cdot P(E_n).$$

• the probability that an event A will occur simultaneously with one of the events $H_1, H_2, \ldots, H_n$ forming a system of mutually exclusive events (hypotheses) is given by the total probability formula:

$$P(A) = \sum_{k=1}^{n} P(H_k)P(A|H_k)$$

where $\sum_{k=1}^{n} P(H_k) = 1$

• the probability $P(H_j|A)$ of the hypothesis H_j after the event A occurred is given by the Bayes' formula:

$$P(H_j|A) = \frac{P(H_j)P(A|H_j)}{\sum\limits_{k=1}^{n} P(H_k)P(A|H_k)}$$

The binomial experiment

• is a statistical experiment that has the following properties:

· the experiment consists of n repeated *trials*

· each trial can result in just two possible outcomes: we call one of these outcomes a *success* and the other, a *failure*

· the probability of success, denoted by p, is the same on every trial.

· the probability of failure, denoted $q = 1 - p$, is the same on every trial

· the trials are independent: the outcome on one trial does not affect the outcome on other trials.

• the binomial probability refers to the probability that a binomial experiment results in exactly k successes out of n trials:

$$P = C_n^k \cdot p^k \cdot q^{n-k}$$

e.g.: flip a coin 6 times, the probability to get 4 heads is:

$$P = C_6^4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^{6-4}$$

• the probability that a binomial experiment results in at least k successes is:

$$P = 1 - \sum_{i=0}^{k-1} C_n^i \cdot p^i \cdot q^{n-i}$$

• the probability that the k-th success is obtained after exactly r trials is:

$$P = C_{r-1}^{k-1} p^k (1-p)^{r-k}, \qquad r \geq k.$$

The multinomial experiment:

• generalizes the binomial experiment:

· now each trial can have k outcomes: $E_1, E_2, \ldots, E_k$

· each of these outcomes have the probabilities $p_1, p_2, \ldots p_k$

· the n trials are again independent.

• the multinomial probability is the probability that E_1 occurs n_1 times, E_2 occurs n_2 times,... E_k occurs n_k times:

$$P = \frac{n!}{n_1!n_2! \cdot \ldots \cdot n_k!} p_1^{n_1} p_2^{n_2} \cdot \ldots \cdot p_k^{n_k}$$

where $n = n_1 + n_2 + \ldots + n_k$

Poisson's scheme:

• let $A_1, A_2, ..., A_n$, be n independent events of an experiment.

• denote by p_i the probability of A_i to occur and by $q_i = 1 - p_i$, $i = \overline{1, n}$ the probability of the complementary event.

• the probability that k events, out of those n, occur is given by the coefficient of X^k in the expression:

$$(p_1 X + q_1) \cdot (p_2 X + q_2) \cdot ... \cdot (p_n X + q_n)$$

Proposed problems

Problem 1. *Consider the numbers* 1, 2, 3,..., n *written in an arbitrary order. What's the probability that the numbers* 1 *and* 2 *are consecutive?*

Problem 2. *Suppose you're a volleyball tournament organizer. There are* 10 *teams signed up for the tournament, and it seems like a good idea for each team to play every other team in a "round robin" setting, before advancing to the playoffs. How many games are possible if each team plays every other team once?*

Problem 3. *Eight students are to be put up in a student residence in three rooms, two of which have three beds and one has two beds. In how many ways can the students be distributed over the three rooms ?*

Problem 4. *How many full houses are possible in poker?*

Problem 5. *A straight consists of five cards with values forming a string of five consecutive values (with no wrap around). For example,* 45678*,* $A2345$ *and* $10JQKA$ *are considered straights, but* $KQA23$ *is not (suits are immaterial for straights.) How many different straights are there in poker?*

Problem 6. *A worker realized* 5 *items of a product. We denote by* $E_i, i = \overline{1, 5}$ *the event: the i-th item is defective. Using set theory describe the following events:*

i) None of the items are defective

ii) At least one of the items is defective

iii) Exactly one of the items is defective

iv) Exactly two are defective

v) At least two items are not defective

vi) At most two items are defective

vii) Assuming $P(E_i) = \frac{1}{10}$, $i = \overline{1,5}$ *estimate the probabilities of the above events.*

Problem 7. *How many rectangles are contained in an* $m \times n$ *checkerboard? Start with the normal chess board.*

Problem 8. *Find the probability that among seven persons:*

a) no two were born on the same day of the week

b) at least two were born on the same day

c) two were born on Sunday and two on Tuesday

Problem 9. *John knows the answers to 1 of the 10 multiple choice questions on the Special Mathematics exam. He has skipped several of the lectures, he must take random guesses for the other nine. Assuming each question has four answers, what is the probability he will get exactly 7 of the last questions right? Every answer worths one point and he needs at least 5 points to pass the exam. What is the probability he will pass?*

Problem 10. *a) An experiment of drawing a random card from an ordinary playing cards deck is done with replacing it back. This was done ten times. Find the probability of getting* 2 *spades* $\spadesuit$, 3 *diamonds* $\diamondsuit$, 3 *clubs* $\clubsuit$ *and* 2 *hearts* $\heartsuit$.

b) The numbers $1, 2, 3, \ldots n$ *are written in random order. What is the probability of having* 1 *and* 2 *on consecutive positions ?*

Problem 11. *At some moment in a backgammon game you have to roll a* 6 *or the sum of the two numbers to be* 6 *to put your opponent's rear checker on the bar. What is the probability of hitting your opponent's checker ? What is the probability of hitting at least twice in* 4 *attempts?*

Problem 12. *Ten shots are fired at a target consisting of an inner circle and two concentric annuli. The probabilities of hitting these regions in one shot are* $0.15, 0.20$ *and* 0.25, *respectively. Find the probability that there will be three hits in the circle, two in the first annulus and two in the second annulus.*

Problem 13. *There are n men and n women (husbands and wives) at some party. What's the probability that at some moment no man dances with his wife? We assume an equal probability in forming a pair, every man has a* $\frac{1}{n}$ *chance to dance with any woman.*

Problem 14. *A question posed in the mid-17th century to Blaise Pascal by a French nobleman and inveterate gambler, the Chevalier de Méré, which marked the birth of probability theory. One of de Méré's favorite bets was that at least one six would appear during a total of four rolls of a die. From past experience, he knew that this gamble paid off more often than not. Then, for a change, he started betting that he would get a double-six on 24 rolls of two dice. However, he soon realized that his old approach to the game was more profitable. He asked his friend Pascal why. Can you give him the answer ?*

Problem 15. *a) A single card is chosen at random from a standard deck of* 52 *playing cards. What is the probability of choosing a king (K) or a club (*$\clubsuit$*)?*

b) A professor gives only two types of exams, "easy" and "hard". You will get a hard exam with probability 0.80*. The probability that the first question on the exam will be marked as difficult is* 0.90 *if the exam is hard and is* 0.15 *otherwise. What is the probability that the first question on your exam is marked as difficult ? What is the probability that your exam is hard given that the first question on the exam is marked as difficult?*

Problem 16. *Two friends decide to meet at* $21:00$ *pm in a restaurant. They decided that who ever reaches the restaurant earlier will wait for the other person for 20 minutes. The restaurant closes at* $23:00$ *pm. What's the probability that those friends do meet?*

Problem 17. *A person wrote 5 letters, sealed them in envelopes and wrote the different addresses randomly on each of them. Find the probability that at least one of the envelopes has the correct address.*

Problem 18. *Find the probability of drawing a king, a queen, a king and a knave in this order, from a deck of* 52 *cards, in four consecutive draws. The cards drawn are not replaced.*

Problem 19. *In USA* 40% *of the registered voters are Republicans,* 45% *are Democrats and* 15% *are Independent. When the voters were asked about increasing military spending* 20% *of the Republicans opposed it,* 65% *of the Democrats opposed it and* 55% *of the Independents opposed it. What is the probability that a randomly selected voter opposes increased military spending ?*

Problem 20. *A telegraphic communications system transmits the signals dot* $\bullet$ *and dash* $^{-}$*. Assume that the statistical properties of the obstacles are such that an average of* 0.4 *of the dots and* 0.25 *of the dashes are changed. Suppose that the ratio between the transmitted dots and the transmitted dashes is* $5:3$*. What is the probability that a received signal will be the same as the transmitted signal if:*

a) the received signal is a dot.

b) the received signal is a dash.