“Science is a differential equation. Relegion is a boundary condition”
Alan Turing

2
Differential equations

A case for Sherlock Holmes

London, 18.30 o’clock.

Watson: Sherlock, tell me again what information do we have already ?

Holmes: Mr. Fox said he was staying at his bedroom window and was watching his neighbour’s house, Mr. Green. The old man was surveilling the henhouse. Since weeks someone steals the eggs of his hens. At 6 o’clock Mr. Fox started to lie in wait. He didn’t see anything unusual. He saw the reverend, Mr. Black, coming along at 9.30 o’clock for a one hour visit. The later declared that everything was alright when he visited him. At 15.00 o’clock came Helen, Mr. Green’s daughter. At 15.45 the front door opened and Helen came out running. She saw Mr. Fox at the window and said something unintelligeble. She was nervous and confused. Mr. Fox met her halfway. *"He's dead!"*, shouted Helen. *"Someone killed him ! He has a head wound!"...*

Watson: Why did she come out after 45 minutes ? It's obvious, she's the murderer !

Holmes: We don't know for sure. She declared that after she saw her father dead she fainted for 45 minutes. Afterwards she came out of the house. The question is: Is she lying or the reverend lies, who claims that Mr. Green was alive at 10.30 o'clock ?

Watson: How are we going to find that out ?

Holmes: I measured the victim's body temperature and the room temperature. Once 2 hours ago and again 5 minutes ago. We'll find the murderer in a few minutes ...

First-order differential equations

We will investigate differential equations of the form:

$$y'(x) = f(x, y(x)), \quad (x, y) \in D_f \subset \mathbb{R}^2.$$

• a function $y = y(x)$ is called a solution (integral, integral curve) of such an equation when it satisfies, together with its derivative, the differential equation.

• the general solution of a first-order differential equation is a one-parameter family of functions $\phi(x, C)$, such that $y = \phi(x, C)$ is a solution. For every C one obtains a particular solution.

• the general integral of the (DE) is a relation:

$$\phi(x, y, C) = 0$$

which defines the general solution implicitly.

• a function $y = y(x)$ is called singular solution when it satisfies the (DE), but it can not be obtained from the general solution formula for a particular choice of the parameter C.

Geometric Interpretation:
By a line-element through the point $P(x_0, y_0)$ we understand a small tangent segment at the solution curve in the point P, given by the slope:

$$m = y'(x_0) = f(x_0, y_0).$$

The set of all line-elements forms the direction field associated with the differential equation.

Example:

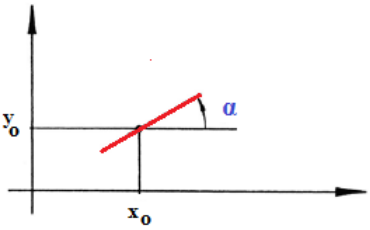
Let's consider the first-order differential equation:

$$y' = x^2 + y^2 := f(x, y)$$

In the point $P(2, 1)$ the line-element has the slope:

$$m = f(2, 1) = 2^2 + 1^2 = 5.$$

which corresponds to the angle α (with the OX axis) such that $\text{tg}(\alpha) = 5$, thus $\alpha = 79°$.



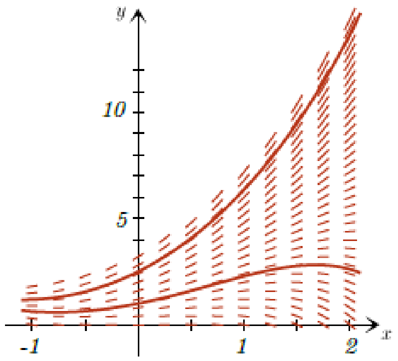
Line-element

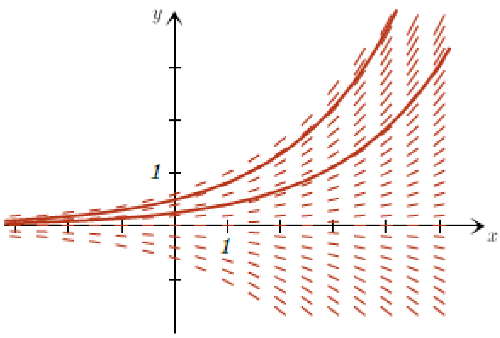
□

The differential equation associates a slope with each point $P(x, y) \in D_f$. If, therefore, a direction field of small line-elements is plotted (see the figures of the next example) the solutions of the differential equation are those curves which "fit in with" this direction field. It means that in each point are tangent to the plotted line-elements. The direction field offers not only an intuitive understanding of the solutions but also the way the solutions behave.

Example:

a) $y' = y - x^2$ b) $y' = \frac{1}{2} y$





The drawn curves show some possible solution curves. In the case

$y' = \frac{1}{2}y$ the function f depends only on y, which also from the figure can be concluded. □

Separable Equations:

$$y' = g(x) \cdot f(y)$$

Algorithm:

- the integration succeeds by separation of variables :

$$\frac{y'}{f(y)} = g(x)$$

⇝ let F be an antiderivative of $\frac{1}{f}$ and G an antiderivative of g, then follows by integration over x:

$$\int \frac{y'(x)}{f(y(x))}dx = \int g(x)dx$$

i.e.:

$$F(y) = G(x) + C.$$

Homogeneous Differential Equations:

$$y' = f\left(\frac{y}{x}\right)$$

Algorithm:

- using the substitution $\frac{y(x)}{x} = z(x)$ we get:

$$xz' + z = f(z)$$

where separation of variables works:

$$\frac{z'}{f(z) - z} = \frac{1}{x}$$

⇝ integration over x provides the general integral:

$$\int \frac{1}{f(z) - z}dz = \ln|x| + C$$

- the case $f(z) - z = 0$ provides the singular solution:

$$y(x) = x \cdot z_0$$

where z_0 is a root for the equation $f(z) - z = 0$.

Differential Equations of the Form:

$$y' = f\left(ax + by + c\right), \quad b \neq 0$$

Algorithm:

- via the substitution:

$$z(x) = ax + by(x) + c$$

follows $z' = a + by'$, thus:

$$y' = \frac{z' - a}{b} = f(z),$$

so, an equivalent DE is:

$$z' = a + bf(z).$$

- This equation is solved by separation of variables.

Example:

Let it be the DE:

$$y' = (2x + 3y)^2 := f(ax + by + c)$$

with $a = 2, b = 3,\ c = 0$ and $f(t) = t^2$. By a substitution:

$$z(x) = 2x + 3y(x)$$

follows $z' = 2 + 3y'$. then $y' = z^2$, provides:

$$z' = 2 + 3z^2.$$

We solve this equation by separation of variables:

$$\int \frac{z'}{2 + 3z^3} dx = \int 1 dx = x + C$$

But:

$$\int \frac{z'(x)}{2 + 3z^3(x)} dx = \frac{1}{2} \int \frac{dz}{1 + \left(\sqrt{\frac{3}{2}} z\right)^2}$$

With a substitution $t = \sqrt{\frac{3}{2}} z$ one gets:

$$\frac{1}{2} \int \frac{dz}{1 + \left(\sqrt{\frac{3}{2}} z\right)^2} = H\left(\sqrt{\frac{3}{2}} z\right) + C$$

where:

$$H(t) = \frac{1}{2} \int \frac{1}{1 + t^2} \sqrt{\frac{2}{3}} dt = \frac{1}{\sqrt{6}} \text{arctg } t$$

Finally:

$$\operatorname{arctg}\left(\sqrt{\frac{3}{2}}z\right) = \sqrt{6}(x + C),$$

and:

$$z(x) = \sqrt{\frac{2}{3}}\operatorname{tg}(\sqrt{6}(x + C)).$$

Therewith the general solution is:

$$y(x) = \frac{z(x) - 2x}{3} = \frac{\sqrt{\frac{2}{3}}\operatorname{tg}(\sqrt{6}(x + C)) - 2x}{3}$$

□

Exact equations:

$$P(x, y)dx + Q(x, y)dy = 0$$

Algorithm:

• an equation of the form:

$$P(x, y) + Q(x, y)y' = 0$$

is sometimes written (for some pseudo-mathematical reasons) in the above form because $y' = \frac{dy}{dx}$

• the point is that when P and Q have continuous partial derivatives and satisfy the condition:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{(exactness condition)}$$

we have nice formula for the general integral of such an equation:

$$\int_{x_0}^{x} P(t, y_0)dt + \int_{y_0}^{y} Q(x, t)dt = c \text{ or } \int_{x_0}^{x} P(t, y)dt + \int_{y_0}^{y} Q(x_0, t)dt = c, \quad x_0, y_0 \in \mathbb{R}.$$

• most of the times such an equation does not satisfy the exactness condition, that's why we'll search for an integrating factor $\mu(x, y)$ such that:

$$\frac{\partial}{\partial y}[\mu(x, y)P(x, y)] = \frac{\partial}{\partial x}[\mu(x, y)Q(x, y)]$$

• it is useful to know that when $\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}$ depends only on x an integrating fractor is:

$$\mu(x) = exp\left(\int_{x_0}^{x} \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} dx\right)$$

and when $\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{P}$ depends only on y an integrating fractor is:

$$\mu(y) = exp\left(-\int_{y_0}^{y} \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{P} dy\right).$$

Linear Homogeneous Differential Equations:

$$y' = f(x)y$$

Algorithm:

- the separation of variables method gives:

$$\frac{y'}{y} = f(x).$$

⇝after an integration over x follows:

$$\ln|x| = \int f(x)dx + C$$

⇝ finally one obtains, for $C > 0$:

$$y(x) = C_1 e^{\int f(x)\ dx},$$

where $C_1 = \ln C \in \mathbb{R}$.

Linear Nonhomogeneous Differential Equations:

$$y' = f(x)y + g(x)$$

Algorithm:

- with the ansatz (variation of constants):

$$y(x) = C(x)e^{\int f(x)\ dx}$$

on gets:

$$C'(x)e^{\int f(x)dx} + C(x)e^{\int f(x)dx}f(x) = f(x)C(x)e^{\int f(x)dx} + g(x),$$

i.e.:

$$C'(x) = g(x)e^{-\int f(x)dx},$$

which implies:

$$y(x) = \left(C + \int g(x)e^{-\int f(x)dx}dx\right)e^{\int f(x)dx}$$

with $C \in \mathbb{R}$ arbitrary.

- the Initial Value Problem:

$$\begin{cases} y'(x) = f(x)y(x) + g(x) \\ y(x_0) = y_0 \end{cases}$$

has the solution:

$$y(x) = \left(y_0 + \int_{x_0}^{x} g(u)e^{-\int_{x_0}^{u} f(t)dt}du\right) \cdot e^{\int_{x_0}^{x} f(t)dt}$$

Bernoulli's Differential Equation:

$$y' + f(x)y = g(x)y^{\alpha}$$

Algorithm:

- for $\alpha > 1$ is $y \equiv 0$ a solution and in the cases $\alpha = 0$, $\alpha = 1$ is the DE linear.
- in the remaining cases we multiply the equation with $y^{-\alpha}$ and hence

$$y^{-\alpha}y' + f(x)y^{1-\alpha} = g(x)$$

$\rightsquigarrow$ with a transform $z(x) = y(x)^{1-\alpha}$ one gets $z'(x) = (1-\alpha)y^{-\alpha}y'$ and further

$$z' + (1-\alpha)f(x)z = (1-\alpha)g(x)$$

$\rightsquigarrow$ this is a linear nonhomogeneous differential equation, and the solving algorithm is known.

Solved problems

How does Sherlock Holmes think ?

According to Newton's law of cooling:

> The rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings provided the temperature difference is small and the nature of radiating surface remains same.

$$T'(t) = -k\big(T(t) - T_C\big)$$

where T is the body temperature (depends on the time variable t), T_C is the room temperature (surrounding) and k is a positive constant. This is a first-order linear differential equation. Normally T_C will not vary too much so we can approximate it as a constant. Thus, we can write:

$$(T(t) - T_C)' = -k(T(t) - T_C)$$

that means:

$$\frac{(T(t) - T_C)'}{T(t) - T_C} = -k$$

Integrating:

$$\ln|T(t) - T_C| = -kt + c, \qquad \text{c constant.}$$

provides:

$$|T(t) - T_C| = e^{-kt} \cdot e^c.$$

Since the body temperature will not be lower than the room temperature

$$|T(t) - T_C| = T(t) - T_C,$$

and the constant c can be eliminated, considering the temperature at the moment of time 0 known, $T(0) = T_0$. Indeed substituting $t = 0$:

$$T(0) - T_C = e^0 \cdot e^c \qquad \Longrightarrow \qquad e^c = T(0) - T_C$$

hence

$$T(t) = T_C + (T_0 - T_C)e^{-kt} \qquad\qquad (*).$$

Sherlock H. has measured the room temperature T_C. He needs now only the value of k. For it he can use the identity

$$\frac{T(t_1) - T_C}{T(t_2) - T_C} = e^{-k(t_1 - t_2)}$$

only if he knows the body temperature in two different moments of time t_1, t_2. The above identity is obtained after one substitutes $t = t_1$ and also $t = t_2$ into $(*)$ and divides the obtained relations. This identity implies

$$k = -\frac{1}{t_1 - t_2} \ln \frac{T(t_1) - T_C}{T(t_2) - T_C} \qquad\qquad (**).$$

For example:

At $t_1 = 16.25$ pm Sherlock H. measured the victim's body temeperature and once again at $t_2 = 18.25$ o'clock. He found the values $T(t_1) = 26.8°C$, respectively $T(t_2) = 26°C$, and $T_C = 25°C$ is the room temperature. Hence:

$$k = -\frac{1}{-2} \ln \frac{26.8 - 25}{26 - 25} = 0.293$$

The normal body temperature varies between $36.3°C$ and $37.4°C$. Sherlock considers the mean temperature:

$$T(t_d) = 37°C$$

at the moment t_d of Mr. Green's death.

With k known he considers further $t_2 = 18.25$ as the 0 moment and $t_1 = t_d$ in the identity $(**)$. Thus $T(0) = 26°C$ and the following relation holds:

$$t_d - 0 = -\frac{1}{k} \ln \frac{T(t_d) - T_C}{T(0) - T_C}$$

$$t_d = -\frac{1}{k} \ln \frac{37 - 25}{26 - 25} \approx -8.4 \quad \text{(hours)}$$

It means that at 10.30 o'clock Mr. Green was not alive.

The reverend, Mr. Black, lies !!!

Problem 1. *Find all possible solutions for the equation:*

$$(1+x)y + (1-y)xy' = 0$$

Solution: Making the assumptions $x \neq 0$ and $y \neq 0$ we can divide by them and separating the variables one obtains:

$$\frac{1+x}{x} = \frac{y-1}{y}y'$$

Integration over x provides:

$$\int \frac{1+x}{x}dx = \int \frac{y-1}{y}y'dx = \int \frac{y-1}{y}dy$$

and further:

$$\ln|x| + x = y - \ln|y| + C$$

thus:

$$xe^x = e^y \frac{1}{y} e^C$$

and finally $e^C = D$:

$$xy = e^{y-x}.$$

Problem 2. *Solve the equation:*

$$x(y^2-1)y' + y(x^2-1) = 0$$

Solution: Again the separation of variables method (for $x \neq 0$ and $y \neq 0$) works fine:

$$\int \frac{1-y^2}{y}dy = \int \frac{1-x^2}{x}dx$$

so

$$\frac{y^2}{2} - \ln y = \ln|x| - \frac{x^2}{2} + C$$

to get the general integral:

$$x^2 + y^2 = 2\ln|xy| + C.$$

Here any of the excluded cases $x = 0$ and $y = 0$ do not represent a solution.

Problem 3. *Solve the equation:*

$$y' - y = xy^5.$$

Which solution curve passes through $P(0, \sqrt{2})$ *?*

Solution: One can recognize a Bernoulli equation with $\alpha = 5$. With the use of the transform $z(x) = y(x)^{1-\alpha} = y(x)^{-4}$ we get the linear equation:

$$z' + 4z = -4x \qquad (*)$$

The solution of this homogeneous differential equation is:

$$z_h = Ce^{-\int 4dx} = De^{-4x}$$

In order to find a particular solution of the nonhomogeneous equation, we try a polynomial. For example:

$$z_p = a + bx.$$

provides:

$$b + 4a + 4bx = -4x,$$

which leads to $b = -1$ und $a = \frac{1}{4}$. In conclusion the solution of the linear DE $(*)$ is:

$$z(x) = z_h(x) + z_p(x) = De^{-4x} - x + \frac{1}{4}$$

Transforming back one obtains:

$$y^4(x) = \frac{4}{4De^{-4x} - 4x + 1}, \quad D \in \mathbb{R}.$$

Another solution is $y \equiv 0$. It can not be obtained from the general solution formula! Hence it is a *singular solution.*

From $y(0) = \sqrt{2}$ follows $4 = \frac{4}{1+4D}$ and $D = 0$. The wanted soution curve is:

$$y(x) = \sqrt[4]{\frac{4}{1-4x}}.$$

Problem 4. *Solve the following equation:*

$$xy' + y = 6x^2, \quad \textit{for} \quad y(1) = 3,$$

and afterwards for the initial data $y(1) = -1$.

Solution: First of all:

$$y' + \frac{y}{x} = 6x,$$

is a nonhomogeneous linear differential equation with $f(x) = -\frac{1}{x}$ and $g(x) = 6x$. The general solution is

$$y(x) = 2x^2 + \frac{C}{x}, \quad C \in \mathbb{R}.$$

Particular solutions: for $y(1) = 3$, one gets $y = 2x^2 + \frac{1}{x}$ and for $y(1) = -1$, follows $y = 2x^2 - \frac{3}{x}$.

Proposed problems

Problem 1. *Solve the equation:*

$$y' = 1 + \sqrt{y - x}$$

making an appropriate change of variable.

Problem 2. *Solve the equation*

$$y' = (-2x + y)^2 - 7, \quad y(0) = 0$$

using an appropriate change of variable.

Problem 3. *Find the general solution of the equation:*

$$yy' = 2y - x.$$

Problem 4. *Solve the equation:*

$$xy' \cos\left(\frac{y}{x}\right) = y \cos\left(\frac{y}{x}\right) - x.$$

Problem 5. *Integrate the following equation:*

$$y' - xy = y$$

Problem 6. *Find the general integral of:*

$$(1 + e^x)yy' - e^x = 0$$

and the particular integral corresponding to $x_0 = 1, y_0 = 1$.

Problem 7. *Integrate the equation:*

$$y^2 + x^2 y' - xyy' = 0$$

Problem 8. *Solve the equation:*

$$(2xy - \sin x)dx + (x^2 - \cos y)dy = 0$$

Problem 9. *Solve the initial value problem:*

$$\frac{1}{y^2} - \frac{2}{x} = \frac{2xy'}{y^3}$$

with initial condition $y(1) = 1$.

Problem 10. *Solve the equation:*

$$2xydx - (x^2 - y^2)dy = 0$$

Problem 11. *Integrate the equation:*

$$(x \sin y + y \cos y)dx + (x \cos y - y \sin y)dy = 0.$$

Problem 12. *Integrate the equation:*

$$xy' + y = y^2 \ln x$$

Problem 13. *Solve the Cauchy problem*

$$\begin{cases} y'' + (y')^2 = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

Problem 14. *Solve the reducible second order equations*

a) $yy'' + (y')^2 = yy'$

b) $xy'' + y' = 8x$

c) $x^2y'' - xy' + y = 0$

d) $xy'' - (x+1)y' + y = 0$

Problem 15. *Solve the initial value problems:*

$$6y' - 2y = xy^4, \qquad y(0) = -2$$

$$xy' + 4y = x^4y^2, \qquad y(2) = -1$$

Bibliography

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