"Is there a better motivation than success ?"

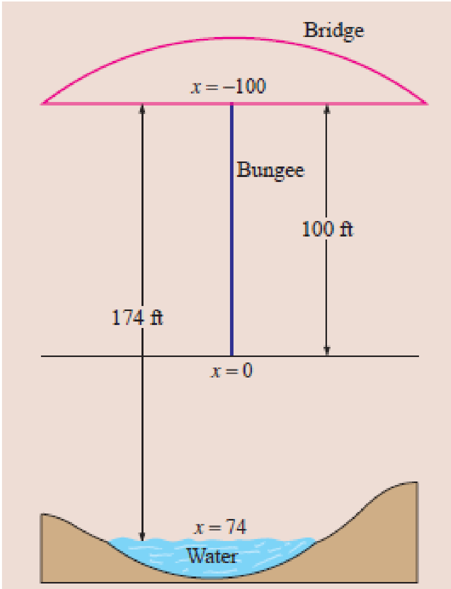
Ion Tiriac

3

Higher order linear equations

Bungee jumping

Bungee jumping is an activity that involves jumping from a tall structure while connected to a large elastic cord. The tall structure is usually a fixed object, such as a building, bridge or crane; but it is also possible to jump from a movable object, such as a hot-air-balloon or helicopter, that has the ability to hover above the ground. The thrill comes from the free-falling and the rebound. When the person jumps, the cord stretches and the jumper flies upwards again as the cord recoils, and continues to oscillate up and down until all the kinetic energy is dissipated.



You decided to try a bungee jumping. Let's suppose that each of the cords you have brought will be tied off so as to be 100 feet long when hanging from the bridge. Call the position at the bottom of the cord 0, and measure the position of your feet below that "natural length" as $x(t)$, where x increases as you go down and is a function of time t. See the figure. Then, at the time you jump, $x(0) = -100$, while if your six-foot frame hits the water head first, at that time $x(t) = 174 - 100 - 6 = 68$. Notice that distance increases as you fall, and so your velocity is positive as you fall and negative when you bounce back up.

Note also that you plan to dive so your head will be six feet below the end of the chord when it stops you. You know that the acceleration due to gravity is a constant, called g, so that the force pulling downwards on your body is mg. You know that when you leap from the bridge, air resistance will increase proportionally to your speed, providing a force in the opposite direction to your motion of about βv, where β is a constant and v is your velocity. Finally, you know that Hooke's law describing the action of springs says that the bungee cord will eventually exert a force on you proportional to its distance past its natural length. Thus, you know that the force of the cord pulling you back from destruction may be expressed as:

$$b(x) = \begin{cases} 0, & x \leq 0 \\ -kx, & x > 0 \end{cases}$$

The number k is called the *spring constant*, and it is where the stiffness of the cord you use influences the equation. For example, if you used the steel cable, then k would be very large, giving a tremendous stopping force very suddenly as you passed the natural length of the cable. This could lead to discomfort, injury, or even a Darwin award. You want to choose the cord with a k value large enough to stop you above or just touching the water, but not too suddenly. Consequently, you are interested in finding the distance you fall below the natural length of the cord as a function of the spring constant. To do that, you must solve the differential equation that we have derived in words above: The force on your body is given by:

$$m \cdot x^{''}(t) = m \cdot g + b(x(t)) - \beta \cdot x'(t)$$

Here mg is your weight, and x is the rate of change of your position below the equilibrium with respect to time: your velocity. The constant β for air resistance depends on a number of things, including whether you wear your skin-tight pink spandex or your skater shorts and XXL T-shirt, but you know that the value today is about 1.0.

This is a nonlinear differential equation, but inside it are two linear differential equations, struggling to get out. When $x < 0$, the equation is:

$$mx''(t) = mg - \beta x'(t)$$

while after you pass the natural length of the cord it is:

$$mx''(t) = mg - kx(t) - \beta x'(t)$$

Remark:

As you can see, knowing a little bit of math is a dangerous thing. We remind you that the assumption that the drag due to air resistance is linear applies only for low speeds. By the time you swoop past the natural length of the cord, that approximation is only wishful thinking, so your actual mileage may vary. Moreover, springs behave nonlinearly in large oscillations, so Hooke's law is only an approximation. Do not trust your life to an approximation made by a man who has been dead for 200 years! Leave bungee jumping to the professionals.

Higher Order Linear Differential Equations

Homogeneous linear equations with constant coefficients:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0$$

where $a_i \in \mathbb{R}$, $i = \overline{1, n}$.

Algorithm:

• write the characteristic polynomial:

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} \ldots + a_{n-1}\lambda + a_n$$

• find the roots, real or complex, of the characteristic equation: $p(\lambda) = 0$.

• for every root we obtain an element of the fundamental system of solutions, according to the following table:

root	generated solution
$\lambda = a$	e^{ax}
$\lambda_1 = \lambda_2 = \ldots = \lambda_k = a$	$e^{ax},\ xe^{ax},\ x^2 e^{ax} \ldots,\ x^{k-1} e^{ax}$
$\lambda = \alpha + \beta i, \quad \lambda = \alpha - \beta i$	$e^{\alpha x} \sin \beta x,\ e^{\alpha x} \cos \beta x$
$\begin{cases} \lambda_1 = \lambda_2 = \ldots = \lambda_k = \alpha + \beta i \\ \lambda_{k+1} = \lambda_{k+2} = \ldots = \lambda_{2k} = \alpha - \beta i \end{cases}$	$\begin{cases} e^{\alpha x} \sin \beta x, xe^{\alpha x} \sin \beta x \ldots, x^{k-1} e^{\alpha x} \sin \beta x \\ e^{\alpha x} \cos \beta x, xe^{\alpha x} \cos \beta x \ldots, x^{k-1} e^{\alpha x} \cos \beta x \end{cases}$
$\lambda = 0$	1
$\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$	$1,\ x,\ x^2,\ \ldots,\ x^{k-1}$

• the general solution of the homogeneous linear equation is:

$$\bar{y}(x) = C_1 \cdot y_1(x) + C_2 \cdot y_2(x) + \ldots + C_n \cdot y_n(x),$$

where $y_1, y_2, \ldots, y_n$ are the n linear independednt solutions of the fundamental system.

Nonhomogeneous linear equations with constant coefficients:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = f(x)$$

Algorithm:

• find the general solution $\bar{y}(x)$ of the attached homogeneous equation

• the general solution for the nonhomogeneous equation will be:

$$y(x) = \bar{y}(x) + y_p(x)$$

where $y_p(x)$ is a particular solution of the nonhomogeneous equation which can be found using the variation of constants method:

↳ if $\bar{y}(x) = C_1 \cdot y_1(x) + C_2 \cdot y_2(x) + \ldots + C_n \cdot y_n(x)$ is a general solution of the attached homogeneous quations we search for a particular solution of the form:

$$y_p(x) = C_1(x) \cdot y_1(x) + C_2(x) \cdot y_2(x) + \ldots + C_n(x) \cdot y_n(x)$$

↳ one finds these functions $C_1(x), C_2(x), \ldots, C_n(x)$ solving the linear system:

$$\begin{cases} C_1'(x) \cdot y_1 + C_2'(x) \cdot y_2 + \ldots + C_n'(x) \cdot y_n = 0 \\ C_1'(x) \cdot y_1' + C_2'(x) \cdot y_2' + \ldots + C_n'(x) \cdot y_n' = 0 \\ C_1'(x) \cdot y_1'' + C_2'(x) \cdot y_2'' + \ldots + C_n'(x) \cdot y_n'' = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_1'(x) \cdot y_1^{(n-2)} + C_2'(x) \cdot y_2^{(n-2)} + \ldots + C_n'(x) \cdot y_n^{(n-2)} = 0 \\ C_1'(x) \cdot y_1^{(n-1)} + C_2'(x) \cdot y_2^{(n-1)} + \ldots + C_n'(x) \cdot y_n^{(n-1)} = f(x) \end{cases}$$

Particular cases:

• for some particular forms of $f(x)$ one can straightforward build a particular solution $y_p(x)$ following the next rules:

1. if $f(x) = e^{\alpha x} P_m(x)$ where P_m is a polynomial of degree m and α is not a root of the characteristic equation, then:

$$y_p(x) = e^{\alpha x} Q_m(x)$$

for Q_m a m-th degree polynomial which has to be determined

2. if $f(x) = e^{\alpha x} P_m(x)$ and α is a multiple root of order k for the characteristic equation, we search for:

$$y_p(x) = x^k e^{\alpha x} Q_m(x)$$

3. if $f(x) = e^{\alpha x} \left[P_m(x) \cos \beta x + Q_m \sin \beta x \right]$ and $\alpha + i\beta$ is not a root for the characteristic equation, we search for:

$$y_p(x) = e^{\alpha x} \left[R_m(x) \cos \beta x + S_m(x) \sin \beta x \right]$$

4. if $f(x) = e^{\alpha x} \left[P_m(x) \cos \beta x + Q_m \sin \beta x \right]$ and $\alpha + i\beta$ is a multiple root of order k for the characteristic equation, we search for:

$$y_p(x) = x^k e^{\alpha x} \left[R_m(x) \cos \beta x + S_m(x) \sin \beta x \right]$$

Solved problems

Problem 1. *Find the general solutions for the next linear differential equations:*

a) $y^{iv} - 5y'' + 4y = 0$,

b) $y''' - 6y'' + 12y' - 8y = 0$,

c) $y^{iv} + 5y'' + 4y = 0$.

Solution: **a)** For the homogeneous equation $y^{iv} - 5y'' + 4y = 0$ we write the characteristic equation:

$$\begin{aligned} r^4 - 5r^2 + 4 &= 0 \overset{r^2=t}{\Rightarrow} t^2 - 5t + 4 = 0 \\ &\Rightarrow (t-1)(t-4) = 0 \end{aligned}$$

thus:

$$\begin{aligned} r^2 &= 1,\ r^2 = 4 \Leftrightarrow \\ r_{1,2} &= \pm 1,\ r_{3,4} = \pm 2. \end{aligned}$$

Since the four roots are real and distinct we build the general solution of the aforementioned equation:

$$y(x) = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}.$$

b) Let's write the characteristic equation:

$$\begin{aligned} r^3 - 6r^2 + 12r - 8 &= 0 \Leftrightarrow (r-2)\left(r^2 - 4r + 4\right) = 0 \\ &\Leftrightarrow (r-2)^3 = 0. \end{aligned}$$

We get a triple root which generates the following form for the general solution:

$$\begin{aligned} y(x) &= c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} \\ &= \left(c_1 + c_2 x + c_3 x^2\right) e^{2x}. \end{aligned}$$

c) For this equation the characteristic equation will be:

$$\begin{aligned} r^4 + 5r^2 + 4 &= 0 \overset{r^2=t}{\Rightarrow} t^2 + 5t + 4 = 0 \\ &\Rightarrow (t+1)(t+4) = 0 \end{aligned}$$

hence:

$$\begin{aligned} r^2 &= -1,\ r^2 = -4 \Leftrightarrow \\ r_{1,2} &= \pm i,\ r_{3,4} = \pm 2i. \end{aligned}$$

The roots are complex and distinct and will generate the general solution:

$$y(x) = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x.$$

Problem 2. *Find the general solutions for the following nonhomogeneous linear equations:*

a) $y'' - y' - 2y = 3e^{2x}$,

b) $y''' + 2y'' - y' - 2y = 2 + e^{x} + \sin x$.

Solution: **a**) We write down the homogeneous equation attached to this problem $y'' - y' - 2y = 0$ which has the characteristic equation:

$$\begin{aligned} r^2 - r - 2 &= 0 \Leftrightarrow \\ (r-2)(r+1) &= 0 \Rightarrow r_1 = 2,\ r_2 = -1. \end{aligned}$$

The general solution of the homogeneous equation will be:

$$\overline{y}(x) = c_1 e^{2x} + c_2 e^{-x}.$$

Let us observe that the function $f(x) = 3e^{2x}$ has $\alpha = 2$ which a a root for the characteristic equation ($r_1 = 2$) thus we have to search for a particular solution of the form:

$$y_p(x) = x^1 \cdot e^{2x} \cdot c.$$

We compute:

$$\begin{aligned} y_p'(x) &= c(1+2x)e^{2x}, \\ y_p''(x) &= 4c(1+x)e^{2x}. \end{aligned}$$

and we substitute in the nonhomogeneous equation to obtain:

$$4c(1+x)e^{2x} - c(1+2x)e^{2x} - 2cxe^{2x} = 3e^{2x} \,|: e^{2x} \Leftrightarrow$$

$$4c(1+x) - c(1+2x) - 2cx = 3$$

$$\Rightarrow c = 1 \Rightarrow y_p(x) = xe^{2x}$$

Eventually, we can display the general solution of the given equation:

$$\begin{aligned} y(x) &= \overline{y}(x) + y_p(x) \\ &= c_1 e^{2x} + c_2 e^{-x} + xe^{2x}. \end{aligned}$$

b) The homogeneous equation $y''' + 2y'' - y' - 2y = 0$ has the characteristic equation

$$\begin{aligned} r^3 + 2r^2 - r - 2 &= 0 \Leftrightarrow (r-1)(r+1)(r+2) = 0 \\ &\Rightarrow r_1 = 1,\ r_2 = -1,\ r_3 = -2, \end{aligned}$$

with the general solution:

$$\overline{y}(x) = c_1 e^{x} + c_2 e^{-x} + c_3 e^{-2x}.$$

The term which is responsible for the nonhomogeneity $f(x) = 2 + e^{x} + \sin x$ has to be decomposed in:

$$2 + e^{x} + \sin x$$

in order to be able to use the particular classes studied in the beginning of these notes:

$$2 = 2 \cdot e^{0 \cdot x} \Rightarrow \alpha = 0$$

$$e^x = e^{1 \cdot x} \Rightarrow \alpha = 1$$

$$\sin x = e^{0 \cdot x} \sin(1 \cdot x) \Rightarrow \alpha = 0 \pm 1 \cdot i.$$

One of these values ($\alpha = 1$) is also a solution for the characteristic equation, thus the particular solution we should search for is:

$$y_p(x) = a + cxe^x + \alpha \cos x + \beta \sin x.$$

We compute now:

$$y_p'(x) = ce^x + cxe^x - \alpha \sin x + \beta \cos x,$$

$$y_p''(x) = 2ce^x + cxe^x - \alpha \cos x - \beta \sin x,$$

$$y_p'''(x) = 3ce^x + cxe^x + \alpha \sin x - \beta \cos x.$$

and we substitute these results in the nonhomogeneous equation to obtain the coefficients:

$$a = -1 \; c = \frac{1}{6}, \; \beta = -\frac{1}{5}, \; \alpha = \frac{1}{10},$$

yielding the particular solution:

$$y_p(x) = -1 + \frac{1}{6}xe^x + \frac{1}{10}\cos x - \frac{1}{5}\sin x,$$

Finally, the general solution of the problem is:

$$\begin{aligned} y(x) &= \overline{y}(x) + y_p(x) = \\ &= c_1 e^x + c_2 e^{-x} + c_3 e^{-2x} - 1 + \frac{1}{6}xe^x + \frac{1}{10}\cos x - \frac{1}{5}\sin x. \end{aligned}$$

Problem 3. *Solve the Cauchy problem:*

$$\begin{cases} y^{iv} - y = x^3 + x \\ y(0) = y'(0) = y''(0) = y'''(0) = 0 \end{cases}$$

Solutieon: The characteristic equation is:

$$r^4 - 1 = 0 \Leftrightarrow \left(r^2 - 1\right)\left(r^2 + 1\right) = 0 \Leftrightarrow (r-1)(r+1)(r-i)(r+i) = 0$$

thus:

$$r_1 = 1, \; r_2 = -1, \; r_3 = i, \; r_4 = -i$$

and will generate the solution:

$$\overline{y}(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x.$$

For the function $f(x) = x^3 + x = e^{0 \cdot x}\left(x^3 + x\right)$ of course $\alpha = 0$ is not a root of the characteristic equation, hence:

$$y_p(x) = ax^3 + bx^2 + cx + d$$

We start to compute:

$$y_p'(x) = 3ax^2 + 2bx + c,$$
$$y_p''(x) = 6ax + 2b,\ \ y_p'''(x) = 6a,\ \ y^{iv}(x) = 0.$$

in order to substitute in the nonhomogeneous equations to obtain:

$$a = -1,\ \ b = 0,\ \ c = -1,\ \ d = 0$$

thus the particular solution

$$\begin{aligned} y_p(x) &= -1 \cdot x^3 + 0 \cdot x^2 - 1 \cdot x + 0 \\ &= -x^3 - x. \end{aligned}$$

Finally the formula for the general solution is:

$$\begin{aligned} y(x) &= \overline{y}(x) + y_p(x) = \\ &= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - x^3 - x. \end{aligned}$$

In order to solve the Cauchy problem we have to make use of the initial data to find the constants c_1, c_2, c_3, c_4 :

$$y(0) = c_1 + c_2 + c_3,$$
$$y'(x) = c_1 e^x - c_2 e^{-x} - c_3 \sin x + c_4 \cos x - 3x^2 - 1 \Rightarrow y'(0) = c_1 - c_2 + c_4 - 1,$$
$$y''(x) = c_1 e^x + c_2 e^{-x} - c_3 \cos x - c_4 \sin x - 6x \Rightarrow y''(0) = c_1 + c_2 - c_3,$$
$$y'''(x) = c_1 e^x - c_2 e^{-x} + c_3 \sin x - c_4 \cos x - 6 \Rightarrow y'''(0) = c_1 - c_2 - c_4 - 6.$$

Using:

$$y(0) = y'(0) = y''(0) = y'''(0) = 0$$

yields:

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 - c_2 + c_4 = 1 \\ c_1 + c_2 - c_3 = 0 \\ c_1 - c_2 - c_4 = 6 \end{cases}$$

with the solution:

$$c_1 = \frac{7}{4},\ \ c_2 = -\frac{7}{4},\ \ c_3 = 0,\ \ c_4 = -\frac{5}{2}.$$

Finally, the Cauchy problem has the solution:

$$y(x) = \frac{7}{4} e^x - \frac{7}{4} e^{-x} - \frac{5}{2} \sin x - x^3 - x,$$

or equivalently:

$$y(x) = \frac{7}{2} \cosh x - \frac{5}{2} \sin x - x^3 - x.$$

Proposed problems

Problem 1. *Find the general solutions for the following linear homogeneous differential equations with constant coefficients:*

a) $64y^{(8)} + 48y^{(6)} + 12y^{(4)} + y^{(2)} = 0,$

b) $y^{iv} - 3y''' + 5y'' - 3y' + 4y = 0.$

Problem 2. *Find the general solutions for the following linear nonhomogeneous differential equations with constant coefficients:*

a) $y'' - 4y' + 4y = 1 + e^x + e^{2x},$

b) $y'' - y = xe^x \sin x,$

c) $y^{iv} - 4y'' = 1,$

d) $y''' - y'' = x.$

f) $y'' - 3y' + 2y = x^2 e^x$

g) $y'' - 4y = e^{2x}(11\cos x - 7\sin x)$

h) $y'' - 2y'' = e^x((4 - 4x)\cos x - (6x + 2)\sin x)$

i) $y''' - y'' + y' - y = \cos x$

j) $y'' - 3y' + 2y = e^{3x}(x^2 + x)$

k) $y'' - 5y' + 6y = 6x^2 - 10x + 2$

Bibliography

[1] Dennis. G. Zill. A First Course in Differential Equations with Modeling Applications, *Brooks/Cole*, 2013.

[2] Octavian Lipovan. Matematici speciale: Ecuatii diferentiale si teoria campurilor, *Editura Politehnica*, 2007.