"Weapons are like money; no one knows the meaning of enough."
Martin Amis

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Systems of first order linear equations

Modelling arms races

Weapons and ammunition recovered during military operations against Taliban militants in South Waziristan in October 2009

The last hundred years have seen numerous dangerous, destabilizing, and expensive arms races. The outbreak of World War I climaxed a rapid buildup of armaments among rival European powers. There was a similar mutual accumulation of conventional arms just prior to World War II. The United States and the Soviet Union engaged in a costly nuclear arms race during the forty years of the Cold War. Stockpiling of ever-more deadly weapons is common today in many parts of the world.

British meteorologist and educator Lewis F. Richardson (1881–1953) developed several mathematical models to analyze the dynamics of arms races, the evolution over time of the process of interaction between countries in their acquisition of weapons. Arms race models generally assume that each nation adjusts its accumulation of weapons in some manner dependent on the size of its own stockpile and the armament levels of the other nations.

Richardson's primary model of a two country arms race is based on mutual fear: *A nation is spurred to increase its arms stockpile at a rate proportional to the level of armament expenditures of its rival.* Richardson's model takes into account internal constraints within a nation that slow down arms buildups: The more a nation is spending on arms, the harder it is to make greater increases, because it becomes increasingly difficult to divert society's resources from basic needs such as food and housing to weapons. Richardson also built into his model other factors driving or slowing down an arms race that are independent of levels of arms expenditures. The mathematical structure of this model is a linked system of two first-orde linear differential equations. If x and y represent the amount of wealth being spent on arms by two nations at time t, then the model has the form:

$$\begin{cases} x' = ay - mx + r \\ y' = bx - ny + s \end{cases}$$

where a, b, m, and n are positive constants while r and s are constants which can be positive or negative. The constants a and b measure mutual fear; the constants m and n represent proportionality factors for the "internal brakes" to further arms increases. Positive values for r and s correspond to underlying factors of ill will or distrust that would persist even if arms expenditures dropped to zero. Negative values for r and s indicate a contribution based on goodwill.

The dynamic behavior of this system of differential equations depends on the relative sizes of ab and mn together with the signs of r and s. *Although the model is a relatively simple one, it allows us to consider several different long-term outcomes*:

• it's possible that two nations might move simultaneously toward mutual disarmament, with x and y each approaching zero.

• a vicious cycle of unbounded increases in x and y is another possible scenario.

• a third eventuality is that the arms expenditures asymptotically approach a stable point (x^*, y^*) regardless of the initial level of arms expenditures.

• in other cases, the eventual outcome depends on the starting point. Figure 1 shows one possible situation with four different initial levels, each of which leads to a "stable outcome", the intersection of the nullclines $x' = 0, y' = 0$.

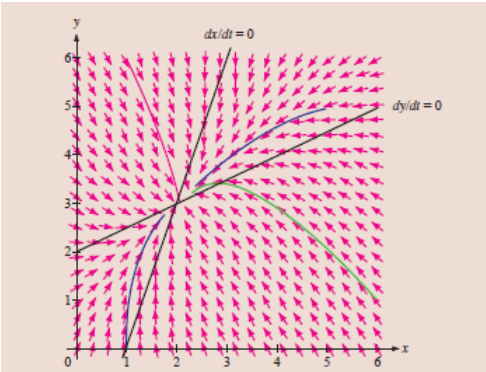


FIGURE 1 Expenditures approaching a stable point

Although "real world" arms races seldom match exactly with Richardson's model, his pioneering work has led to many fruitful applications of differential equation models to problems in international relations and political science. As two leading researchers in the field note "the Richardson arms race model constitutes one of the most important models of arms race phenomena and, at the same time, one of the most influentia formal models in all of the international relations literature." Arms races are not limited to the interaction of nation states. They can take place between a government and a paramilitary terrorist group within its borders as, for example, the Tamil Tigers in Sri Lanka, the Shining Path in Peru, or the Taliban in Afghanistan. Arms phenomena have also been observed between rival urban gangs and between law enforcement agencies and organized crime. The "arms" need not even be weapons. Colleges have engaged in "amenities arms races," often spending millions of dollars on more luxurious dormitories, state- of-the-art athletic facilities, epicurean dining options, and the like, to be more competitive in attracting student applications. Biologists have identified the possibility of evolutionary arms races between and within species as an adaptation in one lineage may change the selection pressure on another lineage, giving rise to a counteradaptation. Most generally, the assumptions represented in a Richardson-type model also characterize many competitions in which each side perceives a need to stay ahead of the other in some mutually important measure.

Systems of linear equations

Homogeneous linear systems with constant coefficients:

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t)\ldots + a_{1n}x_n(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t)\ldots + a_{2n}x_n(t) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n'(t) = a_{n1}x_1(t) + a_{n2}x_2(t)\ldots + a_{nn}x_n(t) \end{cases}$$

where $a_{ij} \in \mathbb{R}$, $i, j = \overline{1,n}$.

- a solution vector on an interval I is any column matrix:

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

whose entries are differentiable functions satisfying the system on the given interval

- any set $X_1, X_2, \ldots X_n$ of n linearly independent solution vectors of the homogeneous system on an interval I is said to be a fundamental set of solutions on the interval

- let $X_1, X_2, \ldots, X_n$ be a fundamental set of solutions of the homogeneous system on the interval I, then the general solution of the system on the interval is:

$$X = c_1X_1 + c_2X_2 + \ldots c_nX_n$$

where $c_i, i = \overline{1,n}$ are arbitrary constants

Algorithm for a two-dimensional system:

- if λ_1, λ_2 are distinct eigenvalues of the matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

then the general solution is:

$$X(t) = c_1V_1e^{\lambda_1 t} + c_2V_2e^{\lambda_2 t}$$

where V_1, V_2 are the linearly independent eigenvectors corresponding to λ_1, λ_2.

- if $\lambda_1 = \lambda_2 \in \mathbb{R}$ are two repeated eigenvalues, the general solution is:

$$X(t) = c_1Ve^{\lambda_1 t} + c_2(tV + W)e^{\lambda_1 t}$$

where V and W satify the equations:

$$(A-\lambda_1 I)V = O_2 \qquad \text{(eigenvector)}$$

$$(A-\lambda_1 I)W = V \qquad \text{(generalized eigenvector)}$$

• if $\lambda_1 = \alpha + i\beta$ is a complex eigenvalue then $\lambda_2 = \alpha - i\beta$ is the other one and let us denote with V the corresponding eigenvector.

$\Longrightarrow$ the general solution is generated by the two linearly independent solutions:

$$X_1(t) = (\mathrm{Re}(V)\cos\beta t - \mathrm{Im}(V)\sin\beta t)e^{\alpha t}$$
$$X_2(t) = (\mathrm{Im}(V)\cos\beta t + \mathrm{Re}(V)\sin\beta t)e^{\alpha t}$$

Nonhomogeneous linear systems with constant coefficients:

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t)\ldots + a_{1n}x_n(t) + f_1(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t)\ldots + a_{2n}x_n(t) + f_2(t) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n'(t) = a_{n1}x_1(t) + a_{n2}x_2(t)\ldots + a_{nn}x_n(t) + f_n(t) \end{cases}$$

where $a_{ij} \in \mathbb{R}$, $i, j = \overline{1,n}$ and $f_i(t), i = \overline{1,n}$ are continuous functions.

• if X_p is a particular solution of the nonhomogeneous system and:

$$X_h = c_1X_1 + c_2X_2 + \ldots c_nX_n$$

is the general solution of the homogeneous system then the general solution of the nonhomogeneous system is:

$$X = X_h + X_p$$

Algorithm for the two-dimensional system:

• if $\Phi(t)$ si a fundamental matrix formed with a fundamental set of solutions of the homogeneous system then the general solution of the nonhomogeneous system is:

$$X(t) = \Phi(t)\cdot C + \Phi(t)\int_0^t \Phi(s)^{-1}F(s)ds$$

where $F(s) = \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix}$ and C is a column matrix of constants.

• for an initial value problem with $X(t_0) = X_0$ the solution is:

$$X(t) = \Phi(t)\Phi^{-1}(t_0)X_0 + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)F(s)ds.$$

Proposed problems

Problem 1. *Consider the particular Richardson arms model:*

$$\begin{cases} x' = y - 3x + 3 \\ y' = 2x - 4y + 8 \end{cases}$$

with the initial conditions $x(0) = 12$ *and* $y(0) = 15$.

What is the long term behaviour of this arms race ?

Problem 2. *Solve the following systems of differential equations:*

i) $\begin{cases} x' = 2x + 3y - 7 \\ y' = -x - 2y + 5 \end{cases}$

ii) $\begin{cases} x' = 2x - y \\ y' = 3x - 2y + 4t \end{cases}$

iii) $X' = \begin{pmatrix} 3 & -5 \\ \frac{3}{4} & -1 \end{pmatrix} X + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\frac{t}{2}}$

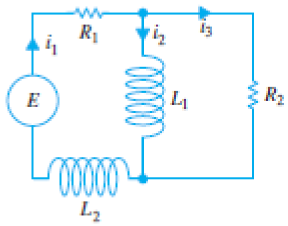
iv) $X' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix} X + \begin{pmatrix} e^{-t} \\ te^{t} \end{pmatrix}$

v) $X' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} X + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^{t}$

iv) $X' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} X + \begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix}$

Problem 3. *The system of differential equations for the currents* $i_1(t)$ *and* $i_2(t)$ *in the electrical network shown in the figure is:*

$$\begin{pmatrix} i_1'(t) \\ i_2'(t) \end{pmatrix} = \begin{pmatrix} -\frac{R_1+R_2}{L_2} & \frac{R_2}{L_2} \\ \frac{R_2}{L_1} & -\frac{R_2}{L_1} \end{pmatrix} \begin{pmatrix} i_1(t) \\ i_2(t) \end{pmatrix} + \begin{pmatrix} \frac{E}{L_2} \\ 0 \end{pmatrix}$$



Use the variation o parameters method to solve the system if $R_1 = 8$, $R_2 = 3$, $L_1 = 1, L_2 = 1$ *and* $E(t) = 100 \sin t$ *with the initial data* $i_1(0) = 0, i_2(0) = 0$.

Bibliography

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[2] Octavian Lipovan. Matematici speciale: Ecuatii diferentiale si teoria campurilor, *Editura Politehnica*, 2007.