"Nature laughs at the difficulties of integration. ."

Piere-Simon Laplace

1
Techniques of Integration

- The integration of rational functions

$R(x) = \frac{P(x)}{Q(x)}$, for P, Q polynomials

reduces to computing one of the following antiderivatives

$$\int x^n dx, \quad \int \frac{M}{(x-x_0)^n} dx \quad \text{or} \quad \int \frac{Ax+B}{(x^2+ax+b)^m} dx$$

since any rational function $\frac{P(x)}{Q(x)}$, with *deg* $P < \deg Q$, can be decomposed as:

$$\begin{aligned}\frac{P(x)}{Q(x)} =& \frac{M_1}{x-x_1} + \frac{M_2}{(x-x_1)^2} + \ldots + \frac{M_{n_1}}{(x-x_1)^{n_1}} \\ &+ \frac{N_1}{x-x_2} + \frac{N_2}{(x-x_2)^2} + \ldots + \frac{N_{n_2}}{(x-x_2)^{n_2}} \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \frac{A_1x+B_1}{x^2+a_1x+b_1} + \frac{A_2x+B_2}{(x^2+a_1x+b_1)^2} + \ldots + \frac{A_{m_1}x+B_{m_1}}{(x^2+a_1x+b_1)^{m_1}} \\ &+ \frac{C_1x+D_1}{x^2+a_2x+b_2} + \frac{C_2x+D_2}{(x^2+a_2x+b_2)^2} + \ldots + \frac{C_{m_2}x+D_{m_2}}{(x^2+a_2x+b_2)^{m_2}} \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

when

$$\begin{aligned}Q(x) =& c(x-x_1)^{n_1}(x-x_2)^{n_2} \cdot \ldots \cdot (x-x_r)^{n_r} \\ &\cdot (x^2+a_1x+b_1)^{m_1}(x^2+a_2x+b_2)^{m_2} \cdot \ldots \cdot (x^2+a_kx+b_k)^{m_k}.\end{aligned}$$

• the first two integrals are elementary, and for the last one we need a change of variables $x + \frac{a}{2} = t$ that will reduce it to

$$\int \frac{t}{(t^2+c^2)^m} dt \quad \text{and} \quad \int \frac{1}{(t^2+c^2)^m} dt$$

$\implies$ the first antiderivative is straightforward using the substitution $t^2 = u$
• in order to find

$$I_m = \int \frac{1}{(t^2+c^2)^m}\, dt, \quad m \in \mathbb{N},\ m \geq 2, c \neq 0$$

one uses the recurrence relation

$$I_m = \frac{1}{2(m-1)c^2} \left(\frac{t}{(t^2+c^2)^{m-1}} + (2m-3) I_{m-1} \right)$$

• Binomial integrals, for $a > 0$ and $ax^n + b > 0$

$$\int x^m (ax^n + b)^p\ dx, \qquad m, n \in p \in \mathbb{Q}.$$

are solved using Chebyshev's substitutions:

$\implies$ for $p \in \mathbb{Z}$, use the substitution $x = t^r$, where $r > 0$ is the least common denominator of m and n

$\implies$ for $\frac{m+1}{n} \in \mathbb{Z}$ use the substitution $ax^n + b = t^r$, where r is the denominator corresponding to p

$\implies$ for $\frac{m+1}{n} + p \in \mathbb{Z}$ use the substitution $\frac{ax^n+b}{x^n} = t^r$ where r is the denominator corresponding to p
• Integrals of type

$$\int R(\sin x, \cos x) dx$$

$\implies$ use $t = \operatorname{tg} \frac{x}{2}$, which implies the following facts $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{1}{1+t^2} dt$
• Integrals of type

$$\int R(x, \sqrt{a^2 - x^2}) dx$$

$\implies$ use $x = a \sin t$ or $x = a \cos t$
• Integrals of type

$$\int R(x, \sqrt{x^2 + a^2})dx$$

$\Longrightarrow$ use $x = a \sinh t = a\frac{e^t - e^{-t}}{2}$, (where sinh is the hyperbolic sine) followed by $e^t = u$

• Integrals of type

$$\int R(x, \sqrt{x^2 - a^2})dx$$

$\Longrightarrow$ use $x = a \cosh t = a\frac{e^t + e^{-t}}{2}$, follwed by $e^t = u$

• Integrals of type

$$\int R(x, \sqrt{ax^2 + bx + c})dx, \quad \text{for } \Delta \neq 0$$

are computed using Euler's substitutions

$\Longrightarrow$ if $a > 0$ use the substitution $\sqrt{ax^2 + bx + c} = \pm x\sqrt{a} \pm t$ (any combination of signs may be chosen)

$\Longrightarrow$ if $c > 0$ use the susbstitution $\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c}$

$\Longrightarrow$ if $\Delta > 0$ use the substitution $\sqrt{ax^2 + bx + c} = \pm t(x - x_1)$, for x_1 a root of $ax^2 + bx + c = 0$

• For definite integrals don't forget to change the integration bounds according to the formulae presented below

$$\int_a^b f(u(x)) \cdot u'(x)\ dx = \int_{u(a)}^{u(b)} f(t)dt$$

using $u(x) = t$ and the first method of integration by susbtitution

$$\int_a^b f(u(x))\ dx = \int_{u(a)}^{u(b)} f(t)\left(u^{-1}(t)\right)'\ dt$$

using $u(x) = t$ and the second method of integration by susbtitution

Solved problems

Problem 1. *Find the antiderivative*

$$\int \frac{1}{\sin x + \cos x + 2}\, dx, \quad \textit{for} \;\; x \in (-\pi, \pi).$$

Solution: Using the substitution $t = \operatorname{tg} \frac{x}{2}$ and the 2nd method of integration by substitution:

$$t = \operatorname{tg} \frac{x}{2}$$

$$\operatorname{arcctg} t = x$$

$$d(\operatorname{arcctg} t) = dx$$

$$(\operatorname{arcctg} t)' dt = dx$$

$$\frac{1}{1+t^2} dt = dx$$

one gets:

$$F(t) = \int \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \frac{1}{1+t^2} dt = \int \frac{2}{t^2+2t+3} dt = 2 \int \frac{1}{(t+1)^2+2} dt = \sqrt{2} \cdot \operatorname{arctg} \frac{t+1}{\sqrt{2}}$$

Eventually

$$\int \frac{1}{\sin x + \cos x + 2}\, dx = F\left(tg \frac{x}{2}\right) + C = \sqrt{2} \cdot \operatorname{arctg} \frac{tg \frac{x}{2} + 1}{\sqrt{2}} + C$$

Problem 2. *Compute*

$$\int \frac{\sqrt{x^2+1}}{x^2}\, dx, \quad \textit{for } x > 0.$$

Solution: The integral can be written equivalently as:

$$\int \frac{\sqrt{x^2+1}}{x^2}\, dx = \int x^{-2} (x^2+1)^{\frac{1}{2}} dx$$

For

$$\frac{-2+1}{2} + \frac{1}{2} \in \mathbb{Z},$$

we apply the third Cebyshev substitution

$$\frac{x^2+1}{x^2} = t^2$$

via the integration by substitution method

$$x = \frac{1}{\sqrt{t^2-1}}$$

$$dx = \frac{-t}{\sqrt{(t^2-1)^3}}dt$$

Thus

$$F(t) = \int \frac{\sqrt{\left(\frac{1}{\sqrt{t^2-1}}\right)^2+1}}{\left(\frac{1}{\sqrt{t^2-1}}\right)^2} \frac{-t}{\sqrt{(t^2-1)^3}}dt = -\int \frac{t^2}{t^2-1}dt$$

$$= -t - \frac{1}{2}\ln\frac{t-1}{t+1} + C$$

As a consequence, one gets

$$\int \frac{\sqrt{x^2+1}}{x^2}\,dx = -\sqrt{\frac{x^2+1}{x^2}} - \frac{1}{2}\ln\frac{\sqrt{\frac{x^2+1}{x^2}}-1}{\sqrt{\frac{x^2+1}{x^2}}+1} + C$$

Problem 3. *Compute the indefinite integral*

$$\int \frac{1}{\sqrt{x^2+x+1}}\,dx.$$

Solution: In order to use Euler's substitutions, let us first observe that $a = 1 > 0$, hence set

$$\sqrt{x^2+x+1} = x + t$$

Then

$$x = \frac{t^2-1}{1-2t}$$

$$dx = \frac{2(t-t^2-1)}{(1-2t)^2}dt$$

and

$$\sqrt{x^2+x+1} = \frac{t-t^2-1}{1-2t}$$

Further, according to the above identities, the associated antiderivative is

$$F(t) = \int \frac{1}{\frac{t-t^2-1}{1-2t}} \frac{2(t-t^2-1)}{(1-2t)^2}dt = 2\int \frac{1}{1-2t}dt = -\ln|1-2t| + C$$

Finally

$$\int \frac{1}{\sqrt{x^2+x+1}}\,dx = -\ln|1+2x-2\sqrt{x^2+x+1}| + C$$

The same integral can be evaluated by the second Euler substitution, since $c = 1 > 0$. For this we set

$$\sqrt{x^2+x+1} = xt + 1$$

and therefore, solving for x

$$x = \frac{2t-1}{1-t^2}$$

The differential element dx becomes

$$dx = \frac{2(t^2 - t + 1)}{(1 - t^2)^2} dt.$$

Plugging all these information into the given integral one obtains

$$F(t) = \int \frac{2}{1 - t^2} dt = \ln \left| \frac{1 + t}{1 - t} \right| + C.$$

Hence

$$\int \frac{1}{\sqrt{x^2 + x + 1}} \, dx = \ln \left| \frac{x - 1 + \sqrt{x^2 + x + 1}}{x + 1 - \sqrt{x^2 + x + 1}} \right| + C,$$

which differs from the previous antiderivative by a constant, most likely.

Problem 4. *Find the integral*

$$J(x) = \int \frac{dx}{(x^2 + 4)^2} \, dx$$

Solution: It is worth recognizing that this integral corresponds to the "dificult case" $I_m = \int \frac{1}{x^2 + c^2} dx$ of the rational functions integration theory:

$$J(x) = I_2(x), \quad \text{with } c = 2.$$

Thus we either use the recurrence relation presented in this handout or we try to obtain it using integration by parts.

Now

$$I_m = \frac{1}{2(m - 1)c^2} \left(\frac{x}{(x^2 + c^2)^{m-1}} + (2m - 3) I_{m-1} \right)$$

hence

$$I_2(x) = \frac{1}{8} \left(\frac{x}{(x^2 + 4)} + I_1(x) \right).$$

Since

$$I_1(x) = \int \frac{1}{x^2 + 4} dx = \frac{1}{2} \text{arctg} \frac{x}{2} + C$$

the conclusion follows:

$$J(x) = \frac{1}{8} \frac{x}{(x^2 + 4)} + \frac{1}{16} \text{arctg} \frac{x}{2} + C.$$

Proposed problems

Problem 1. *Find the antiderivatives*

i) $\displaystyle\int \frac{5x-3}{x^2-2x-3}dx$

ii) $\displaystyle\int \frac{\cos x}{1+\cos x}dx$

iii) $\displaystyle\int \frac{1}{\sqrt{(x-1)^3(x-2)}}dx$

iv) $\displaystyle\int \frac{1}{x+\sqrt{x^2-x+1}}dx$

v) $\displaystyle\int \frac{x^8}{\sqrt{x^3-1}}dx$

vi) $\displaystyle\int \frac{1}{1+\sin^2 x}dx$

vii) $\displaystyle\int \frac{x^{10}}{\sqrt{x^3-1}}dx$

viii) $\displaystyle\int \frac{dx}{x\sqrt{x^2+2x+5}}$

Problem 2. *Compute the definite integrals*

i) $\displaystyle\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\, dx$

ii) $\displaystyle\int_0^1 x^2\sqrt{4-x^2}\, dx$

Problem 3. *Evaluate the following integrals*

a) $\displaystyle\int \cos(15x)\cos(10x)\, dx$

b) $\displaystyle\int \frac{\sin^7 x}{\cos^4 x}\, dx$

c) $\displaystyle\int \frac{x^2+4}{3x^3+4x^2-4x}\, dx$

d) $\displaystyle\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}\, dx$