Integralrechnung: Aufgaben

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Contents

1. Unbestimmtes Integral: Aufgaben

1.1. Grund- oder Stammintegrale (Tabelle 1)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1),$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int e^x\,dx = e^x + C,$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C,$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C,$$

$$\int \frac{dx}{\sin^2 x} = -\cot x + C$$

$$\int \sinh x\,dx = \cosh x + C,$$

$$\int \cosh x\,dx = \sinh x + C$$

$$\int \frac{dx}{\cosh^2 x} = \tanh x + C,$$

$$\int \frac{dx}{\sinh^2 x} = -\coth x + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C \quad (a \neq 0)$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C \quad (a \neq 0,\ |x| \neq |a|)$$

$$\int \frac{x\,dx}{a^2 \pm x^2} = \pm\frac{1}{2}\ln\left|a^2 \pm x^2\right| + C$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C \quad (a > 0)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C \quad (a > 0)$$

$$\int \frac{x\,dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2} + C \quad (a > 0)$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C \quad (a > 0)$$

$$\int \sqrt{x^2 \pm a^2}\,dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{x^2 \pm a^2}\right| \quad (a > 0)$$

1.2. Unbestimmte Integrale (Tabelle 2)

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C, \qquad \int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C \qquad (a \neq 0)$$

2. Einige Regeln

2.1. Binomische Formeln

Binome zweiten Grades

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

Binome höheren Grades

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

2.2. Rechenregeln mit Potenzen

$$a^n \cdot a^m = a^{n+m}, \qquad \frac{a^n}{a^m} = a^{n-m}, \qquad (a^n)^m = a^{n \cdot m}$$
$$a^n \cdot b^n = (a \cdot b)^n, \qquad \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$
$$a^0 = 1, \qquad a^{-n} = \frac{1}{a^n}, \qquad a^n = \frac{1}{a^{-n}}$$

2.3. Rechenregeln mit Logarithmen

$$\log_a (x \cdot y) = \log_a x + \log_a y$$
$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a (x^r) = r \log_a x$$
$$\log_a \frac{1}{x} = -\log_a x$$

2.4. Trigonometrische Formeln

$$\sin^2\alpha + \cos^2\alpha = 1 \tag{1}$$

$$\sin(-\alpha) = -\sin\alpha, \qquad \cos(-\alpha) = \cos\alpha \tag{2}$$

$$\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right), \qquad \cos\alpha = \sin\left(\frac{\pi}{2} - \alpha\right) \tag{3}$$

$$\sin\alpha = \sin(\pi - \alpha), \qquad \cos\alpha = -\cos(\pi - \alpha) \tag{4}$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha, \qquad \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \tag{5}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \tag{6}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{7}$$

$$\sin\alpha\,\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right], \tag{8}$$

$$\cos\alpha\,\cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right] \tag{9}$$

$$\sin\alpha\,\cos\beta = \frac{1}{2}\left[\sin(\alpha - \beta) + \sin(\alpha + \beta)\right] \tag{10}$$

$$\sin^2\alpha = \frac{1}{2}\left(1 - \cos 2\alpha\right), \qquad \cos^2\alpha = \frac{1}{2}\left(1 + \cos 2\alpha\right) \tag{11}$$

$$\sin^3\alpha = \frac{1}{4}\left(3\sin\alpha - \sin(3\alpha)\right), \qquad \cos^3\alpha = \frac{1}{4}\left(\cos(3\alpha) + 3\cos\alpha\right) \tag{12}$$

2.5. Integration von Potenzfunktionen

A1

Beispiel:

$$I = \int \left(5\sqrt[4]{x} + \frac{1}{x\sqrt[3]{x}} - \frac{1}{x^2}\right) dx = \int \left(5x^{1/4} + x^{-4/3} - x^{-2}\right) dx = 4x^{5/4} - \frac{3}{x^{1/3}} - \frac{1}{x} + C$$
$$= 4x\sqrt[4]{x} - \frac{3}{\sqrt[3]{x}} - \frac{1}{x} + C$$

a) $I_1 = \int (2x-3)\,dx, \qquad I_2 = \int (4x^2 - 2x - 3)\,dx, \qquad I_3 = \int (x^2 - 3x)\cdot(2x-1)\,dx$

b) $I_1 = \int \left(3x^2 + 2x - \frac{6}{x^2}\right) dx, \qquad I_2 = \int \left(\frac{4}{x^3} + \frac{6}{x^4} - \frac{10}{x^6}\right)$

c) $I_1 = \int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx, \qquad I_2 = \int \left(\sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x}\right) dx, \qquad I_3 = \int x\,(5\sqrt{x} - 7\sqrt[3]{x})$

d) $I_1 = \int \left(\frac{1}{x\sqrt{x}} - \frac{1}{x^2}\right) dx, \qquad I_2 = \int \left(\frac{1}{\sqrt{x}} - \frac{6}{\sqrt[3]{x}} + \frac{20}{\sqrt[5]{x}}\right) dx$

e) $I_1 = \int \left(\frac{1}{\sqrt[3]{x}} + x\sqrt{x} + 2\right) dx, \qquad I_2 = \int \left(\frac{3}{4\sqrt[4]{x}} - 7x^2\sqrt{x}\right) dx$

f) $I_1 = \int \left(\sqrt{x\sqrt{x}} - \frac{2}{\sqrt{x}}\right) dx, \qquad I_2 = \int \sqrt[3]{x^2\sqrt{x}}\,dx$

A2

a) $I_1 = \int (x^3+1)^2\,dx, \qquad I_2 = \int (x^2-1)^3\,dx, \qquad I_3 = \int x\,(x^4+1)^2\,dx$

b) $I_1 = \int \frac{x^3+3x-1}{x}\,dx, \qquad I_2 = \int \frac{x\,(x^2-1)\,(x+2)}{x+1}\,dx, \qquad I_3 = \int \frac{x^2-3x+4}{\sqrt{x}}\,dx$

c) $I_1 = \int \frac{(x-1)^3}{\sqrt{x}}\,dx, \qquad I_2 = \int \frac{(\sqrt{x}+2)^2}{\sqrt{x}}\,dx, \qquad I_3 = \int \frac{x-1}{\sqrt{x}+1}$

2.6. Elementarintegrale

A3

a) $I_1 = \int (e^x - 2\sin x)\,dx, \qquad I_2 = \int \left(2^x + \sqrt{\frac{1}{x}}\right) dx$

b) $I_1 = \int \left(\cos x + \frac{3}{\sqrt{4-4x^2}}\right) dx, \qquad I_2 = \int \left(\sin x - \frac{5}{\sqrt{9-9x^2}}\right) dx$

c) $I_1 = \int \left(\frac{1}{\sqrt{2-2x^2}} - 3^{-x}\right) dx, \qquad I_2 = \int \left(10^{-x} + \frac{x^2+2}{1+x^2}\right) dx$

A4

Beispiel:

$$\begin{aligned} I &= \int \frac{x^2-3}{2x^2-2}\,dx = \frac{1}{2}\int \frac{x^2-3}{x^2-1}\,dx = \frac{1}{2}\int \frac{x^2-1-2}{x^2-1}\,dx = \frac{1}{2}\left[\int dx - 2\int \frac{dx}{x^2-1}\right] = \\ &= \frac{1}{2}\int dx + \int \frac{dx}{1-x^2} = \frac{x}{2} + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C \end{aligned}$$

$$I_1 = \int \frac{x^2}{2\,(1+x^2)}\,dx, \qquad I_2 = \int \frac{x^2}{x^2-1}\,dx, \qquad I_3 = \int \frac{x^2+2}{x^2-1}\,dx$$

A5

Beispiel:

$$I = \int \frac{5}{2x^2+7}\,dx = \frac{5}{2}\int \frac{1}{x^2+\frac{7}{2}}\,dx = \frac{5}{2}\int \frac{1}{x^2+\left(\sqrt{\frac{7}{2}}\right)^2}\,dx = \frac{5}{2}\int \frac{dx}{x^2+a^2} = \frac{5}{\sqrt{14}}\arctan\frac{\sqrt{2}\,x}{\sqrt{7}} + C,$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C, \qquad a = \sqrt{\frac{7}{2}}$$

$$I_1 = \int \frac{dx}{x^2+4}, \qquad I_2 = \int \frac{dx}{x^2+3}, \qquad I_3 = \int \frac{3}{4\,x^2+3}\,dx$$

2.7. Integration durch Substitution

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du, \qquad u = g(x), \quad du = g'(x)\, dx \tag{13}$$

Das Integral eines Produktes lässt sich immer dann berechnen, wenn ein Faktor $f(g(x))$ eine Funktion einer inneren Funktion und der andere Faktor die Ableitung $g'(x)$ der inneren Funktion ist, sofern für die Funktion f unter Beachtung der Substitution $u = g(x)$ das Integral $\int f(u)\, du$ gelöst werden kann.

A6

Beispiel 1:

$$I = \int (2x+3)^5\, dx, \qquad u = 2x+3, \quad \frac{du}{dx} = 2, \quad dx = \frac{1}{2}\, du$$

$$I = \int (2x+3)^5\, dx = \frac{1}{2} \int u^5\, du = \frac{1}{2}\frac{u^6}{6} + C = \frac{1}{12}(2x+3)^6 + C$$

Beispiel 2:

$$I = \int x^2 (6-x^3)^4\, dx, \qquad u = 6-x^3, \quad \frac{du}{dx} = -3x^2, \quad x^2\, dx = -\frac{du}{3}$$

$$I = \int x^2 (6-x^3)^4\, dx = -\frac{1}{3} \int u^4\, du = -\frac{1}{3}\frac{u^5}{5} + C = -\frac{1}{15}(6-x^3)^5 + C$$

a) $I_1 = \int (2+x)^3\, dx, \qquad I_2 = \int (1+3x)^4\, dx, \qquad I_3 = \int (4-2x)^5\, dx$

b) $I_1 = \int x\,(x^2+1)^2\, dx, \qquad I_2 = \int x\,(2x^2-3)^3\, dx, \qquad I_3 = \int 2x\,(3-x^2)^6\, dx$

c) $I_1 = \int 3x^2\,(1+x^3)^2\, dx, \qquad I_2 = \int 5x^3\,(7-x^4)^4\, dx, \qquad I_3 = \int x^5\,(5-2x^6)^3\, dx$

A7

Beispiel:

$$I = \int (6x-9)(x^2-3x+11)^2\, dx, \qquad u = x^2-3x+11, \quad \frac{du}{dx} = 2x-3, \quad (2x-3)\, dx = du$$

$$I = \int (6x-9)(x^2-3x+11)^2\, dx = 3\int u^2\, du = u^3 + C = (x^2-3x+11)^3 + C$$

$$I_1 = \int (x-1)\,(x^2-2x+1)\,dx$$
$$I_2 = \int (3x^2+2)\,(2x^3+4x-7)\,dx$$
$$I_3 = \int x\,(2x^2-x+1)\left(x^4-\frac{2}{3}x^3+x^2-12\right)^2 dx$$
$$I_4 = \int x^2\,(2x^3-1)\,(x^6-x^3+9)^3\,dx$$

A8

Beispiel:

$$I = \int \frac{x}{2x-6}\,dx = \frac{1}{2}\int \frac{x}{x-3}\,dx = \frac{1}{2}\int \frac{x-3+3}{x-3}\,dx = \frac{1}{2}\int dx + \frac{3}{2}\int \frac{dx}{x-3} =$$
$$= \frac{x}{2} + \frac{3}{2}\ln|x-3| + C$$

$$I_1 = \int \frac{dx}{2+x}, \qquad I_2 = \int \frac{dx}{x-3}, \qquad I_3 = \int \frac{x}{x+4}\,dx$$

A9

Beispiel:

$$I = \int \frac{x\,(9x-4)\,dx}{(3x^3-2x^2+7)^5} = \int \frac{du}{u^5} = -\frac{1}{4u^4} + C = -\frac{1}{4\,(3x^3-2x^2+7)^4} + C$$
$$u = 3x^3-2x^2+7, \qquad \frac{du}{dx} = 9x^2-4x = x\,(9x-4), \qquad dx = \frac{du}{x\,(9x-4)}$$

a) $I_1 = \int \frac{(x+1)\,dx}{(x^2+2x-3)^2}, \qquad I_2 = \int \frac{(3x-2)\,dx}{(3x^2-4x-11)^3}, \qquad I_3 = \int \frac{x\,(2x^2-3)\,dx}{(x^4-3x^2+12)^4}$

A10

Beispiel:

$$I = \int (4-x^2)\,\sqrt[3]{12x-x^3}\,dx = \frac{1}{3}\int u^{1/3}\,du = \frac{1}{4}\,x\,(12-x^2)\,\sqrt[3]{12x-x^3}$$
$$u = 12x-x^3, \qquad \frac{du}{dx} = 12-3x^2 = 3\,(4-x^2), \qquad dx = \frac{du}{3\,(4-x^2)}$$

a) $I_1 = \int \sqrt{x-2}\,dx, \qquad I_2 = \int \sqrt{3x-6}\,dx, \qquad I_3 = \int \sqrt[3]{2x+7}$

b) $I_1 = \int \sqrt[3]{(4-3x)^2}\,dx, \qquad I_2 = \int x\,\sqrt{x^2+3}\,dx, \qquad I_3 = \int (6x^2-4)\,\sqrt[4]{x^3-2x+11}\,dx,$

A11

Beispiel 1:

$$I = \int e^{7-3x}\,dx = -\frac{1}{3}\int e^u\,du = -\frac{1}{3}e^u + C$$

$$u = 7-3x, \qquad \frac{du}{dx} = -3, \qquad dx = -\frac{1}{3}\,du$$

Beispiel 2:

$$I = \int \frac{dx}{4e^x-2} = \frac{1}{2}\int\frac{dx}{2e^x-1} = \frac{1}{2}\int\frac{du}{u(u+1)} = \frac{1}{2}\int\left(\frac{1}{u}-\frac{1}{u+1}\right)du =$$

$$= \frac{1}{2}\left(\int\frac{du}{u}-\frac{du}{u+1}\right) = \frac{1}{2}\left(\int\frac{du}{u}-\int\frac{dv}{v}\right) = \frac{1}{2}(\ln u - \ln v) + C_1 = \frac{1}{2}(\ln u - \ln(u+1)) + C_1$$

$$= \frac{1}{2}(\ln(2e^x-1) - \ln(2e^x)) + C_1 = \frac{1}{2}\ln(2e^x-1) - \frac{1}{2}(\ln 2 + \ln(e^x)) + C_1 =$$

$$= \frac{1}{2}\ln(2e^x-1) - \frac{1}{2}x + C_2, C_2 = -\frac{1}{2}\ln 2 + C_1$$

$$u = 2e^x-1, \qquad \frac{du}{dx} = 2e^x = u+1, \qquad dx = \frac{du}{u+1}$$

$$v = u+1, \qquad dv = du$$

a) $I_1 = \int e^{2x}\,dx, \qquad I_2 = \int e^{3-2x}\,dx, \qquad I_3 = \int e^{mx+n}\,dx \quad (m,\, n \neq 0)$

b) $I_1 = \int x\,e^{x^2}\,dx, \qquad I_2 = \int x^2\,e^{2x^3}\,dx, \qquad I_3 = \int (x+1)\,e^{x^2+2x-3}\,dx$

c) $I_1 = \int \frac{dx}{e^x-1}, \qquad I_2 = \int \frac{dx}{2e^x-3}, \qquad I_3 = \int \frac{e^x\,dx}{3e^x+5}$

A12

a) $I_1 = \int_0^{\pi} \sin(2x)\,\cos\,x\,dx, \qquad I_2 = \int_0^{\pi/2} \sin(3x)\,\cos\,x\,dx$

b) $I_1 = \int_0^{\pi/4} \sin(2x)\,\sin\,x\,dx, \qquad I_2 = \int_0^{\pi/3} \sin(4x)\,\sin(2x)\,dx$

c) $I_1 = \int_0^{\pi/4} \cos(2x)\,\cos\,x\,dx, \qquad I_2 = \int_0^{\pi/6} \cos(4x)\,\cos(2x)\,dx$

A13

$$I_a = \int \sin^2(ax)\,dx, \qquad I_b = \int \cos^2(ax)\,dx$$

A14

Beispiel:

$$\begin{aligned} I &= \int \frac{\sin(4x)}{12+8\cos(4x)}\,dx = \frac{1}{4}\int \frac{\sin(4x)}{3+2\cos(4x)}\,dx = \frac{1}{4}\int \frac{\sin(4x)}{u}\left(-\frac{1}{8}\right)\frac{du}{\sin(4x)} = \\ &= -\frac{1}{32}\int \frac{du}{u} = -\frac{1}{32}\ln|u| + C = -\frac{1}{32}\ln|3+2\cos(4x)| + C, \\ u &= 3+2\cos(4x), \quad \frac{du}{dx} = -2\cdot\sin(4x)\cdot 4 = -8\sin(4x), \quad dx = -\frac{1}{8}\,\frac{du}{\sin(4x)} \end{aligned}$$

a) $I_1 = \int \frac{\cos x}{1+\sin x}\,dx, \qquad I_2 = \int \frac{\sin x}{1+\cos x}\,dx, \qquad I_3 = \int \frac{\cos x}{3+2\sin x}\,dx$

b) $I_1 = \int \frac{\sin x}{7-3\cos x}\,dx, \qquad I_2 = \int \frac{\sin(2x)}{1+\cos(2x)}\,dx, \qquad I_3 = \int \frac{\cos(3x)}{4+2\sin(3x)}\,dx$

A15

Beispiel 1:

$$\begin{aligned} I &= \int \sin^3 x\cos x\,dx = \int u^3\cos x\,\frac{du}{\cos x} = \int u^3\,du = \frac{u^4}{4} + C = \frac{1}{4}\sin^4 x + C \\ u &= \sin x, \quad \frac{du}{dx} = \cos x, \quad dx = \frac{du}{\cos x} \end{aligned}$$

Beispiel 2:

$$\begin{aligned} I &= \int \cos^6(2x)\,\sin(4x)\,dx = 2\int \cos^6(2x)\,\sin(2x)\,\cos(2x)\,dx = 2\int \cos^7(2x)\,\sin(2x)\,dx = \\ &= 2\int u^7\,\sin(2x)\left(-\frac{du}{2\,\sin(2x)}\right) = -\int u^7\,du = -\frac{u^8}{8} + C = -\frac{1}{8}\cos^8(2x) + C \\ u &= \cos(2x), \quad \frac{du}{dx} = -2\,\sin(2x), \quad dx = -\frac{du}{2\,\sin(2x)} \end{aligned}$$

a) $I_1 = \int \sin^5 x\,\cos x\,dx, \qquad I_2 = \int \cos^5 x\,\sin x\,dx, \qquad I_3 = \int \sin^7 x\,\cos x\,dx$

b) $I_1 = \int \sin^3 x\,\sin(2x)\,dx, \qquad I_2 = \int \cos^5 x\,\sin(2x)\,dx, \qquad I_3 = \int \sin^8(2x)\,\sin(4x)\,dx$

Hinweis: $\sin(2x) = 2\,\sin x\,\cos x$

A16

Beispiel:

$$I = \int \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{(\sin x)'}{1+\sin^2 x}\,dx = \int \frac{\cos x}{1+u^2}\cdot\frac{du}{\cos x} = \int \frac{du}{1+u^2} = \arctan u + C = $$

$$= \arctan(\sin x) + C,$$

$$u = \sin x, \quad \frac{du}{dx} = \cos x, \quad dx = \frac{du}{\cos x}$$

<u>Grundintegral:</u> $\int \frac{dx}{1+x^2} = \arctan x + C$

a) $I_1 = \int \frac{\sin x}{1+\cos^2 x}\,dx, \qquad I_2 = \int \frac{\cos x}{4+\sin^2 x}\,dx, \qquad I_3 = \int \frac{3\,\sin x}{6+2\,\cos^2 x}\,dx$

b) $I_1 = \int \frac{\sin(2x)}{1+\cos^2(2x)}\,dx, \qquad I_2 = \int \frac{\cos(2x)}{4+\sin^2(2x)}\,dx, \qquad I_3 = \int \frac{\sin(3x)}{2+5\cos^2(3x)}\,dx$

A17

Beispiel:

$$I = \int \frac{dx}{1+\sqrt{x}} = 2\int \frac{u\,du}{1+u} = 2\int \frac{(u+1-1)\,du}{1+u} = 2\int\left(1-\frac{1}{1+u}\right)du =$$

$$= 2\left(\int du - \int \frac{dv}{v}\right) = 2\,(u-\ln|v|) + C = 2\,(u-\ln|1+u|) + C =$$

$$= 2\sqrt{x} - 2\,\ln|1+\sqrt{x}| + C$$

$$u = \sqrt{x}, \quad \frac{du}{dx} = \frac{1}{2}\,x^{-1/2} = \frac{1}{2\sqrt{x}}, \quad dx = 2\sqrt{x}\,du = 2\,u\,du$$

$$v = 1+u, \quad dv = du$$

a) $I_1 = \int \frac{dx}{2+\sqrt{x+1}}, \qquad I_2 = \int \frac{dx}{4+\sqrt{x-3}}, \qquad I_3 = \int \frac{dx}{1+\sqrt{3x-2}}$

b) $I_1 = \int \frac{\sqrt{x}\,dx}{x+4}, \qquad I_2 = \int \frac{\sqrt{x}\,dx}{x+a^2}, \qquad I_3 = \int \frac{x\sqrt{x}\,dx}{x+9}$

A18

a) $I_1 = \int \frac{\sqrt[3]{x}+1}{\sqrt{x}}\,dx, \qquad I_2 = \int \frac{\sqrt{x+1}}{\sqrt[3]{x+1}}\,dx, \qquad I_3 = \int \frac{\sqrt{2x+5}}{\sqrt[4]{2x+5}}\,dx$

b) $I_1 = \int \frac{4\sqrt{x+2}-3}{\sqrt[3]{x+2}}\,dx, \qquad I_2 = \int \frac{dx}{x\sqrt{x+1}}, \qquad I_3 = \int \frac{dx}{x\sqrt{2x-3}}$

A19

$$I_1 = \int x\cdot\sqrt{x+2}\,dx, \qquad I_2 = \int x\cdot\sqrt{x+5}\,dx, \qquad I_3 = \int x^2\cdot\sqrt{2x-5}\,dx.$$

2.8. Partielle Integration

$$\int u\,v'\,dx = u\,v - \int u'\,v\,dx \tag{14}$$

A20

Beispiel 1:

$$I = \int x\,\ln x\,dx = \frac{x^2}{2}\,\ln x - \int \frac{1}{x}\,\frac{x^2}{2}\,dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + C,$$

$$u = \ln x, \quad u' = \frac{1}{x}, \qquad v' = x, \quad v = \int x\,dx = \frac{x^2}{2}$$

Beispiel 2:

$$I = \int \ln^2\,x dx = x\,\ln^2\,x - 2\int \ln x\,dx = x\,\ln^2\,x - 2\,x\,\ln x + 2x + C$$

$$\int \ln x\,dx = x\,(\ln x - 1) + C$$

$$u = \ln^2\,x, \quad u' = \frac{2}{x}\,\ln x, \qquad v' = 1, \quad v = x$$

Beispiel 2:

$$I = \int \sqrt{x}\,\ln\,x\,dx = \frac{2}{3}\,x^{3/2}\,\ln\,x - \frac{2}{3}\int x^{1/2}\,dx = \frac{2}{3}\,x^{3/2}\left(\ln\,x - \frac{2}{3}\right) + C =$$

$$= \frac{2}{3}\,x\,\sqrt{x}\left(\ln\,x - \frac{2}{3}\right) + C$$

$$u = \ln\,x, \quad u' = \frac{1}{x}, \qquad v' = \sqrt{x}, \quad v = \int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2}{3}\,x^{3/2}$$

a) $I_1 = \int \ln^3\,x\,dx, \qquad I_2 = \int \ln^4\,dx, \qquad I_3 = \int \ln^5\,dx$

b) $I_1 = \int x^2\,\ln\,x\,dx, \qquad I_2 = \int x^3\,\ln\,x\,dx, \qquad I_3 = \int x\,\ln\,(x^2)\,dx$

c) $I_1 = \int (x+1)\,\ln\,x\,dx, \qquad I_2 = \int (x+1)^2\,\ln\,x\,dx, \qquad I_3 = \int x\,\ln\,(x+1)$

d) $I_1 = \int \sqrt{x}\,\ln(\sqrt{x})\,dx, \qquad I_2 = \int \sqrt[3]{x}\,\ln x\,dx, \qquad I_3 = \int \sqrt[5]{x^2}\,\ln x\,dx$

A21

$$I_1 = \int \frac{\ln\,x}{x}\,dx, \qquad I_2 = \int \frac{\ln\,x}{x^2}\,dx, \qquad I_3 = \int \frac{\ln\,x}{x^3}\,dx$$

A22

Beispiel 1:

$$\begin{aligned} I &= \int x\,e^x dx = x\,e^x - \int e^x\,dx = (x-1)\,e^x + C, \\ u &= x, \quad u' = 1, \qquad v' = e^x, \quad v = \int e^x\,dx = e^x \end{aligned}$$

$$I_1 = \int x^2\,e^x\,dx, \qquad I_2 = \int x^3\,e^x\,dx, \qquad I_3 = \int x^4\,e^x\,dx$$

A23

$a)\quad I_1 = \int x\,\sin x\,dx, \qquad I_2 = \int x\,\cos x\,dx, \qquad I_3 = \int x\,\sin(2x)\,dx, \qquad I_4 = \int x\,\cos(2x)\,dx$

$b)\quad I_1 = \int x^2\,\sin x\,dx, \qquad I_2 = \int x^2\,\cos x\,dx, \qquad I_3 = \int x^2\,\cos(2x)\,dx, \qquad I_4 = \int x^2\,\sin(3x)\,dx$

3. Anwendungen der Integralrechnung

3.1. Flächen zwischen Kurven: Aufgaben

3.1.1. Eingeschlossene Flächen

Berechnen Sie die von den Kurven $f(x)$ und $g(x)$ eingeschlossene Fläche

A24 $f(x) = x^2 - 2, \quad g(x) = 2$

A25 $f(x) = 2x - x^2, \quad g(x) = -3$

A26 $f(x) = x^2 - 2x, \quad g(x) = x$

3.1.2. Flächen zwischen Kurven in einem Intervall

Welche Fläche schließen die Kurven im Intervall I

A27 $f(x) = x^2 - 2, \quad g(x) = 2, \quad I = [-2, 3]$

A28 $f(x) = x^2 - 2x, \quad g(x) = x, \quad I = [-1, 3]$

A29 $f(x) = x^2 - 2x, \quad g(x) = x + 1, \quad I = [-1, 2]$

4. Unbestimmtes Integral: Lösungen

4.1. Integration von Potenzfunktionen

L1

a) $I_1 = \int (2x-3)\,dx = x^2 - 3x + C, \qquad I_2 = \int (4x^2 - 2x - 3)\,dx = \frac{4}{3}x^3 - x^2 - 3x + C$

$$I_3 = \int (x^2 - 3x)\cdot(2x-1)\,dx = \frac{1}{2}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + C$$

b) $I_1 = \int \left(3x^2 + 2x - \frac{6}{x^2}\right) dx = x^3 + x^2 + \frac{6}{x} + C,$

$$I_2 = \int \left(\frac{4}{x^3} + \frac{6}{x^4} - \frac{10}{x^6}\right) = -\frac{2}{x^2}\left(1 + \frac{1}{x} - \frac{1}{x^3}\right) + C$$

c) $I_1 = \int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \int \left(x^{1/2} + x^{-1/2}\right) dx = \frac{2}{3}\sqrt{x}\,(3+x) + C,$

$$I_2 = \int \left(\sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x}\right) dx = \frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + \frac{5}{6}x^{6/5} + C = \frac{2}{3}x\sqrt{x} + \frac{3}{4}x\sqrt[3]{x} + \frac{5}{6}x\sqrt[5]{x} + C,$$

$$I_3 = \int x\left(5\sqrt{x} - 7\sqrt[3]{x}\right) = 2x^{5/2} - 3x^{7/3} + C = x^2\,(2\sqrt{x} - 3\sqrt[3]{x}) + C$$

d) $I_1 = \int \left(\frac{1}{x\sqrt{x}} - \frac{1}{x^2}\right) dx = -\frac{2}{\sqrt{x}} + \frac{1}{x} + C,$

$$I_2 = \int \left(\frac{1}{\sqrt{x}} - \frac{6}{\sqrt[3]{x}} + \frac{20}{\sqrt[5]{x}}\right) dx = 2\sqrt{x} - 9\,x^{2/3} + 25\,x^{4/5} + C$$

e) $I_1 = \int \left(\frac{1}{\sqrt[3]{x}} + x\sqrt{x} + 2\right) dx = \frac{3}{2}x^{2/3} + \frac{2}{5}x^{5/2} + 2x + C = \frac{3}{2}\sqrt[3]{x^2} + \frac{2}{5}x^2\sqrt{x} + 2x + C,$

$$I_2 = \int \left(\frac{3}{4\sqrt[4]{x}} - 7x^2\sqrt{x}\right) dx = x^{3/4} - 2\,x^{7/2} + C = x^{3/4} - 2\,x^3\sqrt{x} + C$$

f) $I_1 = \int \left(\sqrt{x\sqrt{x}} - \frac{2}{\sqrt{x}}\right) dx = \int \left(x^{3/4} - 2\,x^{-1/2}\right) dx = \frac{4}{7}x\cdot x^{3/4} - 4\sqrt{x} = \frac{4}{7}x\sqrt[4]{x^3} - 4\sqrt{x}$

$$I_2 = \int \sqrt[3]{x^2\sqrt{x}}\,dx = \frac{6}{11}x\cdot x^{5/6} = \frac{6}{11}x\sqrt[6]{x^5}$$

L2

$$a)\quad I_1 = \int (x^3+1)^2\,dx = \frac{1}{7}\,x^7 + \frac{1}{2}\,x^4 + x + C,$$

$$I_2 = \int (x^2-1)^3\,dx = \frac{1}{7}\,x^7 - \frac{3}{5}\,x^5 + x^3 - x + C,$$

$$I_3 = \int x\,(x^4+1)^2\,dx = \frac{1}{10}\,x^{10} + \frac{1}{3}\,x^6 + \frac{1}{2}\,x^2$$

$$b)\quad I_1 = \int \frac{x^3+3x-1}{x}\,dx = \frac{1}{3}\,x^3 + 3x - \ln x + C,$$

$$I_2 = \int \frac{x\,(x^2-1)\,(x+2)}{x+1}\,dx = \int x\,(x-1)\,(x+2) = \frac{1}{4}\,x^4 + \frac{1}{3}\,x^3 - x^2 + C,$$

$$I_3 = \int \frac{x^2-3x+4}{\sqrt{x}}\,dx = \frac{2}{5}\,\sqrt{x}\,(20-5x+x^2) + C,$$

$$c)\quad I_1 = \int \frac{(x-1)^3}{\sqrt{x}}\,dx = \frac{2}{35}\,\sqrt{x}\,(-35+35x-21x^2+5x^3) + C,$$

$$I_2 = \int \frac{(\sqrt{x}+2)^2}{\sqrt{x}}\,dx = \frac{2}{3}\,x^{3/2} + 4x + 8\,\sqrt{x} + C = \frac{2}{3}\,x\,\sqrt{x} + 4x + 8\,\sqrt{x} + C,$$

$$I_3 = \int \frac{x-1}{\sqrt{x}+1}\,dx = \int (\sqrt{x}-1)\,dx = \frac{2}{3}\,x^{3/2} - x + C$$

4.2. Elementarintegrale

L3

$$a)\quad I_1 = \int (e^x - 2\,\sin x)\,dx = e^x + 2\,\cos x + C,$$

$$I_2 = \int \left(2^x + \sqrt{\frac{1}{x}}\right)dx = \frac{2^x}{\ln 2} + 2\,\sqrt{x} + C$$

$$b)\quad I_1 = \int \left(\cos x + \frac{3}{\sqrt{4-4x^2}}\right)dx = \int \left(\cos x + \frac{3}{2\,\sqrt{1-x^2}}\right)dx = \sin x + \frac{3}{2}\,\arcsin x + C,$$

$$I_2 = \int \left(\sin x - \frac{5}{\sqrt{9-9x^2}}\right)dx = \int \left(\sin x - \frac{5}{3\,\sqrt{1-x^2}}\right)dx = -\cos x - \frac{5}{3}\,\arcsin x + C$$

$$c)\quad I_1 = \int \left(\frac{1}{\sqrt{2-2x^2}} - 3^{-x}\right)dx = \frac{1}{\sqrt{2}}\,\arcsin x + \frac{3^{-x}}{\ln 3} + C,$$

$$I_2 = \int \left(10^{-x} + \frac{x^2+2}{1+x^2}\right)dx = -\frac{10^{-x}}{\ln(10)} + x + \arctan x + C$$

L4

$$I_1 = \int \frac{x^2}{2\,(1+x^2)}\,dx = \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx = \frac{x}{2} - \frac{1}{2}\arctan x + C$$

$$I_2 = \int \frac{x^2}{x^2-1}\,dx = \int \frac{x^2-1+1}{x^2-1}\,dx = x - \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$

$$I_3 = \int \frac{x^2+2}{x^2-1}\,dx = \int \frac{(x^2-1+1)+2}{x^2-1}\,dx = x - \frac{3}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$

L5

$$I_1 = \int \frac{dx}{x^2+4} = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C,$$

$$I_2 = \int \frac{dx}{x^2+3} = \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C,$$

$$I_3 = \int \frac{3\,dx}{4\,x^2+3} = \frac{\sqrt{3}}{2}\arctan\left(\frac{2x}{\sqrt{3}}\right) + C$$

4.3. Integration durch Substitution

L6

$$a)\quad I_1 = \int (2+x)^3\, dx = \frac{(2+x)^4}{4} + C, \qquad u = 2+x,$$

$$I_2 = \int (1+3x)^4\, dx = \frac{(1+3x)^5}{15} + C, \qquad u = 1+3x,$$

$$I_3 = \int (4-2x)^5\, dx = -\frac{(4-2x)^6}{12} + C, \qquad u = 4-2x$$

$$b)\quad I_1 = \int x\,(x^2+1)^2\, dx = \frac{1}{2}\int u^2\, du = \frac{(x^2+1)^3}{6} + C, \qquad u = x^2+1,$$

$$I_2 = \int x\,(2x^2-3)^3\, dx, \qquad u = 2x^2-3,$$

$$I_3 = \int 2x\,(3-x^2)^6\, dx, \qquad u = 3-x^2$$

$$c)\quad I_1 = \int 3x^2\,(1+x^3)^2\, dx, \qquad u = 1+x^3,$$

$$I_2 = \int 5x^3\,(7-x^4)^4\, dx, \qquad u = 7-x^4,$$

$$I_3 = \int x^5\,(5-2x^6)^3\, dx, \qquad u = 5-2x^6$$

L7

$$I_1 = \int (x-1)\,(x^2-2x+1)\, dx = \frac{1}{4}\,(x^2-2x+1)^2 + C, \qquad u = x^2-2x+1, \qquad dx = \frac{du}{2(x-1)}$$

$$I_2 = \int (3x^2+2)\,(2x^3+4x-7)\, dx = \frac{1}{4}\,(2x^3+4x-7)^2 + C,$$

$$u = 2x^3+4x-7, \qquad dx = \frac{du}{2(3x^2+2)}$$

$$I_3 = \int x\,(2x^2-x+1)\left(x^4-\frac{2}{3}\,x^3+x^2-12\right)^2 dx = \frac{1}{6}\left(x^4-\frac{2}{3}\,x^3+x^2-12\right)^3,$$

$$u = x^4-\frac{2}{3}\,x^3+x^2-12, \qquad dx = \frac{du}{2x\,(2x^2-x+1)}$$

$$I_4 = \int x^2\,(2x^3-1)\,(x^6-x^3+9)^3\, dx = \frac{1}{12}\,(x^6-x^3+9)^4,$$

$$u = x^6-x^3+9, \qquad dx = \frac{du}{3x^2\,(2x^3-1)}$$

L8

$$I_1 = \int \frac{dx}{2+x} = \ln|2+x| + C,$$

$$I_2 = \int \frac{dx}{x-3} = \ln|x-3| + C,$$

$$I_3 = \int \frac{x}{x+4}\,dx = \int \frac{x+4-4}{x+4}\,dx = \int dx - 4\int \frac{dx}{x+4} = x - 4\ln|x+4| + C$$

L9

a) $$I_1 = \int \frac{(x+1)\,dx}{(x^2+2x-3)^2} = -\frac{1}{2\,(x^2+2x-3)} + C, \qquad u = x^2+2x-3,$$

$$I_2 = \int \frac{(3x-2)\,dx}{(3x^2-4x-11)^3} = -\frac{1}{4\,(3x^2-4x-11)^2} + C, \qquad u = 3x^2-4x-11,$$

$$I_3 = \int \frac{x\,(2x^2-3)\,dx}{(x^4-3x^2+12)^4} = -\frac{1}{6\,(x^4-3x^2+12)^3} + C, \qquad u = x^4-3x^2+12$$

L10

a) $$I_1 = \int \sqrt{x-2}\,dx = \frac{2}{3}\,(x-2)^{3/2} + C = \frac{2}{3}\,(x-2)\,\sqrt{x-2} + C, \qquad u = x-2,$$

$$I_2 = \int \sqrt{3x-6}\,dx = \frac{2}{3}\,(x-2)\,\sqrt{3x-6} + C, \qquad u = 3x-6,$$

$$I_3 = \int \sqrt[3]{2x+7} = \frac{3}{8}\,(2x+7)^{4/3} + C = \frac{3}{8}\,(2x+7)\,\sqrt[3]{2x+7} + C, \qquad u = 2x+7$$

b) $$I_1 = \int \sqrt[3]{(4-3x)^2}\,dx = \frac{1}{5}\,(3x-4)\,\sqrt[3]{(4-3x)^2} + C, \qquad u = 4-3x,$$

$$I_2 = \int x\,\sqrt{x^2+3}\,dx = \frac{1}{3}\,(x^2+3)^{3/2} + C = \frac{1}{3}\,(x^2+3)\,\sqrt{x^2+3} + C, \qquad u = x^2+3,$$

$$I_3 = \int (6x^2-4)\,\sqrt[4]{x^3-2x+11}\,dx = \frac{8}{5}\,(x^3-2x+11)^{5/4} + C =$$

$$= \frac{8}{5}\,(x^3-2x+11)\,\sqrt[4]{x^3-2x+11} + C, \qquad u = x^3-2x+11$$

L11

$$a)\quad I_1 = \int e^{2x}\,dx \;=\; \frac{1}{2}\,e^{2x} + C, \qquad u = 2x,$$

$$I_2 = \int e^{3-2x}\,dx \;=\; -\frac{1}{2}\,e^{3-2x} + C, \qquad u = 3-2x,$$

$$I_3 = \int e^{mx+n}\,dx \;=\; \frac{1}{m}\,e^{mx+n} + C, \qquad u = mx+n \quad (m,\; n \neq 0)$$

$$b)\quad I_1 = \int x\,e^{x^2}\,dx \;=\; \frac{1}{2}\,e^{x^2} + C, \qquad u = x^2,$$

$$I_2 = \int x^2\,e^{2x^3}\,dx \;=\; \frac{1}{6}\,e^{2x^3} + C, \qquad u = 2x^3,$$

$$I_3 = \int (x+1)\,e^{x^2+2x-3}\,dx \;=\; \frac{1}{2}\,e^{x^2+2x-3} + C, \qquad u = x^2+2x-3$$

$$c)\quad I_1 = \int \frac{dx}{e^x-1} \;=\; \ln|e^x-1| \;-\; \ln(e^x) + C \;=$$

$$=\; \ln|e^x-1| \;-\; x + C, \qquad \ln(e^x) = x, \qquad u = e^x - 1,$$

$$I_2 = \int \frac{dx}{2e^x-3} \;=\; \int \frac{du}{u\,(u+3)} \;=\; \frac{1}{3}\int\left(\frac{1}{u} - \frac{1}{u+3}\right)du \;=$$

$$= \frac{1}{3}\left(\int \frac{du}{u} - \frac{du}{u+3}\right) \;=\; \frac{1}{3}\,(\ln|u| - \ln|u+3|) + C_1,$$

$$= \frac{1}{3}\,\ln|2e^x-3| - \frac{1}{3}\,\ln(2e^x) + C_1 \;=\; \frac{1}{3}\,\ln|2e^x-3| - \frac{1}{3}\,(\ln 2 + \ln(e^x)) + C_1 \;=$$

$$= \frac{1}{3}\,\ln|2e^x-3| - \frac{1}{3}\,x - \frac{1}{3}\,\ln 2 + C_1 \;=\; \frac{1}{3}\,\ln|2e^x-3| - \frac{1}{3}\,x + C_2,$$

$$u = 2e^x - 3, \qquad \frac{du}{dx} = 2e^x, \qquad dx = \frac{du}{2e^x} = \frac{du}{u+3}, \qquad \ln(e^x) = x, \qquad C_2 = C_1 - \frac{1}{3}\,\ln 2,$$

$$I_3 = \int \frac{e^x\,dx}{3e^x+5} \;=\; \frac{1}{3}\int \frac{du}{u} \;=\; \frac{1}{3}\,\ln(u) + C \;=\; \frac{1}{3}\,\ln(3e^x+5) + C,$$

$$u = 3e^x + 5, \qquad \frac{du}{dx} = 3e^x, \qquad e^x\,dx = \frac{du}{3}$$

L12

$$a) \quad I_1 = \int_0^{\pi} \sin(2x)\, \cos\, x\, dx$$

$$\sin\, \alpha\, \cos\, \beta = \frac{1}{2}\, [\sin(\alpha - \beta) + \sin(\alpha + \beta)] \qquad (Eq.\ (10),\ \text{Seite } 3)$$

$$\sin(2x)\, \cos\, x = \frac{1}{2}\, (\sin\, x + \sin(3x))$$

$$I_1 = \int_0^{\pi} \sin(2x)\, \cos\, x\, dx = \frac{1}{2} \int_0^{\pi} (\sin\, x + \sin(3x))\, dx =$$

$$-\frac{1}{2} \left[\frac{1}{3}\, \cos(3x) + \cos\, x\right]_0^{\pi} = \frac{4}{3}$$

$$I_2 = \int_0^{\pi/2} \sin(3x)\, \cos\, x\, dx = \frac{1}{2} \int_0^{\pi/2} (\sin(2x) + \sin(4x))\, dx =$$

$$= -\frac{1}{4} \left[\frac{1}{2}\, \cos(4x) + \cos(2x)\right]_0^{\pi/2} = \frac{1}{2}$$

$$\text{Variante } 2: \qquad \sin^3 \alpha = \frac{1}{4}\, (3\, \sin \alpha - \sin(3\alpha)) \qquad (Eq.\ (12),\ \text{Seite } 3)$$

$$\sin(3x) = 3 \sin\, x - 4 \sin^3 x,$$

$$I_2 = \int_0^{\pi/2} \sin(3x)\, \cos\, x\, dx = \int_0^{\pi/2} (3 \sin\, x - 4 \sin^3 x)\, \cos\, x\, dx =$$

$$= 3 \int_0^{\pi/2} \sin\, x\, \cos\, x\, dx - 4 \int_0^{\pi/2} \sin^3 x\, \cos\, x\, dx =$$

$$= \left[-\frac{3}{2}\, \cos^2 x - \sin^4 x\right]_0^{\pi/2} = \frac{4}{3}$$

$$b) \quad I_1 = \int_0^{\pi/4} \sin(2x)\, \sin\, x\, dx =$$

$$1) \quad = \frac{2}{3} \left[\sin^3 x\right]_0^{\pi/4} = \frac{1}{3\sqrt{2}},$$

$$2) \quad = \frac{1}{6}\, [3\, \sin x - \sin(3x)]_0^{\pi/4} = \frac{1}{3\sqrt{2}},$$

$$I_2 = \int_0^{\pi/3} \sin(4x)\, \sin(2x)\, dx = \frac{1}{4} \left[\sin(2x) - \frac{1}{3}\, \sin(6x)\right]_0^{\pi/3} = \frac{\sqrt{3}}{8}$$

$$c)\quad I_1 = \int_0^{\pi/4} \cos(2x)\,\cos\,x\,dx = \frac{1}{2}\left[\sin\,x + \frac{1}{3}\,\sin(3x)\right]_0^{\pi/4} = \frac{\sqrt{2}}{3}$$

$$I_2 = \int_0^{\pi/6} \cos(4x)\,\cos(2x)\,dx = \frac{1}{4}\left[\sin(2x) + \frac{1}{3}\,\sin(6x)\right]_0^{\pi/6} = \frac{\sqrt{3}}{8}$$

L13

$$I_a = \int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\,\sin(2ax) + C, \qquad I_b = \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\,\sin(2ax) + C$$

L14

$$a)\quad I_1 = \int \frac{\cos x}{1+\sin x}\,dx = \ln|1+\sin x| + C, \qquad u = 1 + \sin x,$$

$$I_2 = \int \frac{\sin x}{1+\cos x}\,dx = -\ln|1+\cos x| + C, \qquad u = 1 + \cos x,$$

$$I_3 = \int \frac{\cos x}{3+2\sin x}\,dx = \frac{1}{2}\,\ln|3+2\sin x| + C, \qquad u = 3 + 2\sin x$$

$$b)\quad I_1 = \int \frac{\sin x}{7-3\cos x}\,dx = \frac{1}{3}\,\ln|7-3\cos x| + C, \qquad u = 7 - 3\cos x,$$

$$I_2 = \int \frac{\sin(2x)}{1+\cos(2x)}\,dx = -\frac{1}{2}\,\ln|1+\cos(2x)| + C, \qquad u = 1 + \cos(2x),$$

$$I_3 = \int \frac{\cos(3x)}{4+2\sin(3x)}\,dx = \frac{1}{6}\,\ln|2+\sin(3x)| + C_1 = \frac{1}{6}\,\ln|4+2\sin(3x)| + C_2,$$

$$C_1 = C_2 + \frac{\ln 2}{6}, \qquad u = 2 + \sin(3x)$$

L15

$$a)\quad I_1 = \int \sin^5 x\,\cos x\,dx = \frac{1}{6}\,\sin^6 x + C, \qquad u = \sin x,$$

$$I_2 = \int \cos^5 x\,\sin x\,dx = -\frac{1}{6}\,\cos^6 x + C, \qquad u = \cos x,$$

$$I_3 = \int \sin^7 x\,\cos x\,dx = \frac{1}{8}\,\sin^8 x + C, \qquad u = \sin x$$

$$b)\quad I_1 = \int \sin^3 x\,\sin(2x)\,dx = 2\int \sin^4 x\,\cos x\,dx = \frac{2}{5}\,\sin^5 x + C, \qquad u = \sin x,$$

$$I_2 = \int \cos^5 x\,\sin(2x)\,dx = 2\int \cos^6 x\,\sin x\,dx = -\frac{2}{7}\,\cos^7 x + C, \qquad u = \cos x,$$

$$I_3 = \int \sin^8(2x)\,\sin(4x)\,dx = 2\int \sin^9(2x)\,\cos(2x)\,dx = \frac{1}{10}\,\sin^{10}(2x) + C, \qquad u = \sin(2x)$$

L16

$$a)\quad I_1 = \int \frac{\sin x}{1+\cos^2\, x}\, dx \;=\; -\arctan(\cos x) \,+\, C, \qquad u \;=\; \cos x,$$

$$I_2 = \int \frac{\cos x}{4+\sin^2\, x}\, dx \;=\; \frac{1}{2}\;\arctan\left(\frac{1}{2}\;\sin x\right) \,+\, C, \qquad u \;=\; \sin x,$$

$$I_3 = \int \frac{3\;\sin x}{6+2\;\cos^2\, x}\, dx = -\,\frac{\sqrt{3}}{2}\;\arctan\left(\frac{1}{\sqrt{3}}\;\cos x\right) \,+\, C, \qquad u \;=\; \cos x,$$

$$b)\quad I_1 = \int \frac{\sin(2x)}{1+\cos^2(2x)}\, dx \;=\; -\,\frac{1}{2}\;\arctan(\cos(2x)), \qquad u \;=\; \cos(2x),$$

$$I_2 = \int \frac{\cos(2x)}{4+\sin^2(2x)}\, dx \;=\; \frac{1}{4}\;\arctan\left(\frac{1}{2}\;\sin(2x)\right) \,+\, C, \qquad u \;=\; \sin(2x),$$

$$I_3 = \int \frac{\sin(3x)}{2+5\cos^2(3x)}\, dx \;=\; -\,\frac{1}{3\;\sqrt{10}}\;\arctan\left(\sqrt{\frac{5}{2}}\;\cos(3x)\right) \,+\, C, \qquad u \;=\; \cos(3x)$$

L17

$$a)\quad I_1 = \int \frac{dx}{2+\;\sqrt{x+1}} \;=\; 2\,\sqrt{x+1} - 4\ln\left|2+\;\sqrt{x+1}\right| \,+\, C, \qquad u \;=\; \sqrt{x+1},$$

$$I_2 = \int \frac{dx}{4+\;\sqrt{x-3}} \;=\; 2\,\sqrt{x-3} - 8\ln\left|\;\sqrt{x-3}+4\right| \,+\, C, \qquad u \;=\; \sqrt{x-3},$$

$$I_3 = \int \frac{dx}{1+\;\sqrt{3x-2}} \;=\; \frac{2}{3}\,\sqrt{3x-2} \;-\; \frac{2}{3}\ln\left|1+\;\sqrt{3x-2}\right| \,+\, C, \qquad u \;=\; \sqrt{3x-2}$$

$$b)\quad I_1 = \int \frac{\sqrt{x}\,dx}{x+4} \;=\; 2\,\sqrt{x} - 4\arctan\left(\frac{\sqrt{x}}{2}\right) \,+\, C, \qquad u \;=\; \sqrt{x},$$

$$I_2 = \int \frac{\sqrt{x}\,dx}{x+a^2}, \qquad u = \sqrt{x}, \quad du = \frac{dx}{2\,\sqrt{x}},$$

$$I_2 = \int \frac{\sqrt{x}\,dx}{x+a^2} = 2\int \frac{u^2\,du}{a^2+u^2} = 2\int \frac{(u^2+a^2-a^2)\,du}{a^2+u^2} = 2u - a^2$$

$$= 2\left(\int du - a^2\int \frac{du}{a^2+u^2}\right) = 2u - 2a^2\int \frac{du}{a^2+u^2}$$

$$I_2 = \int \frac{\sqrt{x}\,dx}{x+a^2} \;=\; 2\,\sqrt{x} - 2a\arctan\left(\frac{\sqrt{x}}{a}\right) \,+\, C, \qquad u \;=\; \sqrt{x},$$

$$I_3 = \int \frac{x\,\sqrt{x}\,dx}{x+9} \;=\; \int \frac{(x+9-9)\;\sqrt{x}}{x+9}\,dx \;=\; \int \sqrt{x}\,dx \;-\; 9\int \frac{\sqrt{x}\,dx}{x+9} \;=$$

$$= \frac{2}{3}\,x\;\sqrt{x} \;-\; 18\;\sqrt{x} \;+\; 54\;\arctan\left(\frac{\sqrt{x}}{3}\right)$$

L18

a) $$I_1 = \int \frac{\sqrt[3]{x}+1}{\sqrt{x}}\,dx = \frac{5}{6}\,x^{5/6} + 2\sqrt{x} + C = \frac{5}{6}\sqrt[6]{x^5} + 2\sqrt{x} + C,$$

$$I_2 = \int \frac{\sqrt{x+1}}{\sqrt[3]{x+1}}\,dx = \frac{6}{7}\,(x+1)^{7/6} + C = \frac{6}{7}\,(x+1)\sqrt[6]{x+1} + C, \qquad u = \sqrt{x+1},$$

$$I_3 = \int \frac{\sqrt{2x+5}}{\sqrt[4]{2x+5}}\,dx = \frac{2}{5}\,(2x+5)^{5/4} + C = \frac{2}{5}\,(2x+5)\sqrt[4]{2x+5} + C, \qquad u = 2x+5$$

b) $$I_1 = \int \frac{4\sqrt{x+2}-3}{\sqrt[3]{x+2}}\,dx = \frac{24}{7}\,(x+2)^{7/6} - \frac{9}{2}\,(x+2)^{2/3} + C =$$

$$= \frac{24}{7}\,(x+2)\sqrt[6]{x+2} - \frac{9}{2}\sqrt[3]{(x+2)^2} + C, \qquad u = \sqrt{x+2},$$

$$I_2 = \int \frac{dx}{x\,\sqrt{x+1}} = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C \qquad u = \sqrt{x+1}$$

$$I_3 = \int \frac{dx}{x\,\sqrt{2x-3}} = \frac{2}{\sqrt{3}}\,\arctan\left(\frac{\sqrt{2x-3}}{\sqrt{3}}\right) + C, \qquad u = \sqrt{2x-3}$$

L19

$$I_1 = \int x\cdot\sqrt{x+2}\,dx = \frac{2}{15}\,(x+2)^{3/2}\,(3x-4) + C,$$

$$I_2 = \int x\cdot\sqrt{x+5}\,dx = \frac{2}{15}\,(x+5)^{3/2}\,(3x-10) + C,$$

$$I_3 = \int x^2\cdot\sqrt{2x-5}\,dx = \frac{1}{21}\,(2x-5)^{3/2}\,(3x^2+6x+10) + C.$$

4.4. Partielle Integration

L20

$$a)\quad I_1 = \int \ln^3 x\,dx = x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x + C =$$

$$= x(\ln^3 x - 3\ln^2 x + 6\ln x - 6) + C$$

$$u = \ln^3 x,\quad u' = \frac{3}{x}\qquad v' = 1,\quad v = x$$

$$\int \ln^2 x\,dx = x(\ln^2 x - 2\ln x + 2) + C,$$ aus dem Beispiel zur Aufgabe

$$I_2 = \int \ln^4 x\,dx = x(\ln^4 x - 4\ln^3 x + 12\ln^2 x - 24\ln x + 24) + C,$$

$$u = \ln^4 x,\qquad v' = 1$$

$$I_3 = \int \ln^5 x\,dx = x(\ln^5 x - 5\ln^4 x + 20\ln^3 x - 60\ln^2 x + 120\ln x - 120) + C,$$

$$u = \ln^5 x,\qquad v' = 1$$

$$b)\quad I_1 = \int x^2\ln x\,dx = \frac{x^3}{9}(3\ln x - 1) + C,\qquad u = \ln x,\quad v' = x^2,$$

$$I_2 = \int x^3\ln x\,dx = \frac{x^4}{16}(4\ln x - 1) + C,\qquad u = \ln x,\quad v' = x^3,$$

$$I_3 = \int x\ln(x^2)\,dx =$$

$$c)\quad I_1 = \int (x+1)\ln x\,dx = \int \ln x\,dx + \int x\ln x\,dx = x\ln x\left(\frac{x}{2}+1\right) - \frac{x^2}{4} - x + C,$$

$$I_2 = \int (x+1)^2\ln x\,dx = \int (x^2+2x+1)\ln x\,dx =$$

$$= \int x^2\ln x\,dx + 2\int x\ln x\,dx + \int \ln x\,dx =$$

$$= x\ln x\left(\frac{x^2}{3} + x + 1\right) - \frac{x^3}{9} - \frac{x^2}{2} - x + C,$$

$$I_3 = \int x\ln(x+1) = (x+1)\ln(x+1)\left(\frac{1}{2}(x+1) - 1\right) - \frac{x^2}{4} + \frac{x}{2} + \frac{3}{4} + C$$

$$d)\quad I_1 = \int \sqrt{x}\,\ln(\sqrt{x})\,dx = \frac{1}{3}x^{3/2}\left(\ln x - \frac{2}{3}\right) + C = \frac{1}{3}x\sqrt{x}\left(\ln x - \frac{2}{3}\right) + C$$

$$I_2 = \int \sqrt[3]{x}\,\ln x\,dx = \frac{3}{4}x^{4/3}\left(\ln x - \frac{3}{4}\right) + C = \frac{3}{4}x\sqrt[3]{x}\left(\ln x - \frac{3}{4}\right) + C$$

$$I_3 = \int \sqrt[5]{x^2}\,\ln x\,dx = \frac{5}{7}x^{7/5}\left(\ln x - \frac{5}{7}\right) + C = \frac{5}{7}x\sqrt[5]{x^2}\left(\ln x - \frac{5}{7}\right) + C$$

L21

$$I_1 = \int \frac{\ln x}{x}\,dx = \frac{1}{2}\ln^2 x + C,$$
$$I_2 = \int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}(\ln x + 1) + C,$$
$$I_3 = \int \frac{\ln x}{x^3}\,dx = -\frac{1}{4x^2}(2\ln x + 1) + C$$

L22

$$I_1 = \int x^2 e^x\,dx = e^x\left(x^2 - 2x + 2\right) + C,$$
$$I_2 = \int x^3 e^x\,dx = e^x\left(x^3 - 3x^2 + 6x - 6\right) + C,$$
$$I_3 = \int x^4 e^x\,dx = e^x\left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) + C$$

L23

$$a)\quad I_1 = \int x\sin x\,dx = \sin x - x\cos x + C,$$
$$I_2 = \int x\cos x\,dx = \cos x + x\sin x + C,$$
$$I_3 = \int x\sin(2x)\,dx = \frac{1}{4}\sin(2x) - \frac{x}{2}\cos(2x) + C,$$
$$I_4 = \int x\cos(2x)\,dx = \frac{1}{4}\cos(2x) + \frac{x}{2}\sin(2x) + C$$

$$b)\quad I_1 = \int x^2\sin x\,dx = -x^2\cos x + 2\cos x + 2x\sin x + C,$$
$$I_2 = \int x^2\cos x\,dx = x^2\sin x - 2\sin x + 2x\cos x + C,$$
$$I_3 = \int x^2\cos(2x)\,dx = \frac{1}{2}\sin(2x)\left(x^2 - \frac{1}{2}\right) + \frac{x}{2}\cos(2x) + C,$$
$$I_4 = \int x^2\sin(3x)\,dx = \frac{1}{3}\cos(3x)\left(-x^2 + \frac{2}{9}\right) + \frac{2}{9}x\sin(3x) + C$$

5. Anwendungen der Integralrechnung

5.1. Flächen zwischen Kurven: Lösungen

Berechnen Sie die von den Kurven $f(x)$ und $g(x)$ eingeschlossene Fläche

L24

$$f(x) = x^2 - 2, \qquad g(x) = 2$$

$$f(x) = g(x) : \qquad x^2 - 2 = 2 \quad \Leftrightarrow \quad x^2 = 4 \quad \Rightarrow \quad x_1 = -2, \quad x_2 = 2,$$

$$A = \int_{-2}^{2} (2 - (x^2 - 2))\, dx = 2 \int_{0}^{2} (4 - x^2)\, dx = \frac{32}{3} \simeq 10.67$$

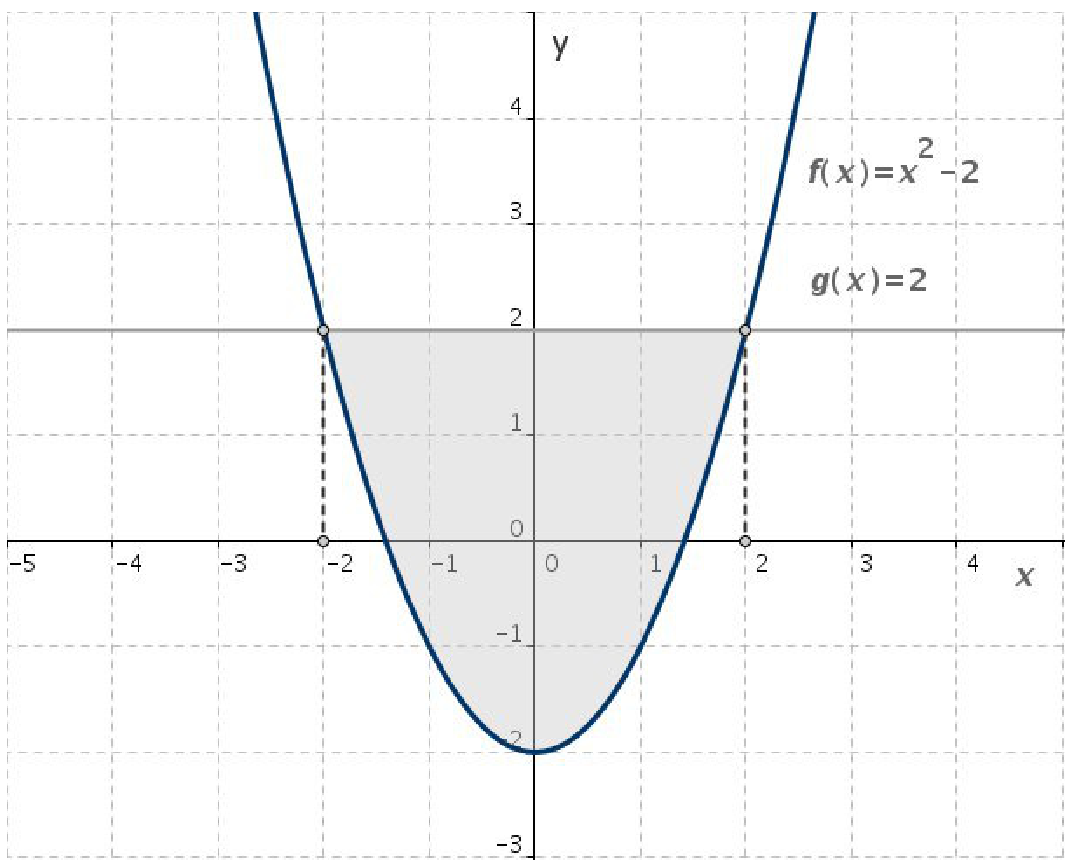

Fig. 1: Die Fläche zwischen der Parabel $f(x) = x^2 - 2$ und der Geraden $g(x) = 2$

L25

$$f(x) = 2x - x^2, \quad g(x) = -3$$

$$f(x) = g(x): \quad 2x - x^2 = -3 \quad \Leftrightarrow \quad x^2 - 2x - 3 = 0 \quad \Rightarrow \quad x_1 = -1, \quad x_2 = 3,$$

$$A = \int_{-1}^{3} (2x - x^2 + 3)\, dx = \frac{32}{3} \simeq 10.67$$

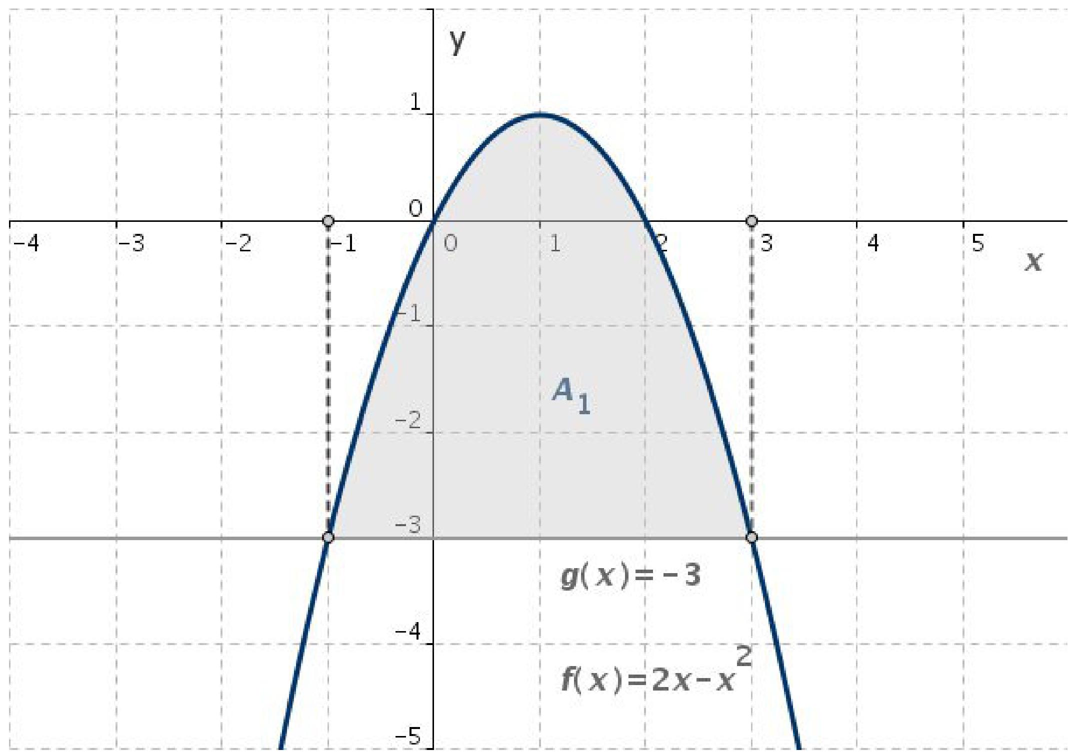

Fig. 2: Die Fläche zwischen der Parabel $f(x) = 2x - x^2$ und der Geraden $g(x) = -3$

L26

$$f(x) = x^2 - 2x, \qquad g(x) = x$$

$$f(x) = g(x) : \qquad x^2 - 2x = x \quad \Leftrightarrow \quad x^2 - 3x = x\,(x - 3) = 0 \quad \Rightarrow \quad x_1 = 0, \quad x_2 = 3,$$

$$A = \int_0^3 (g(x) - f(x))\,dx = \int_0^3 (3x - x^2)\,dx = \frac{9}{2} = 4.5$$

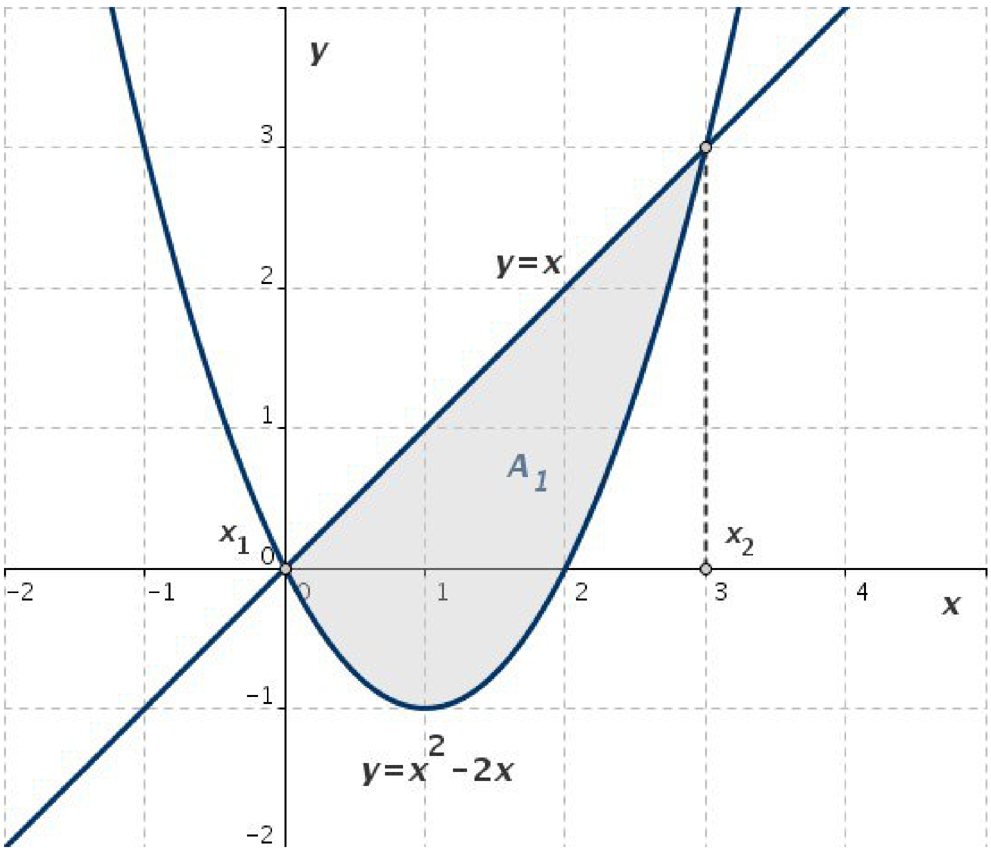

Fig. 3: Die Fläche zwischen der Parabel $f(x) = x^2 - 2x$ und der Geraden $g(x) = x$

L27

$$f(x) = x^2 - 2, \quad g(x) = 2, \quad I = [-2,\ 3]$$

$$f(x) = g(x): \quad x^2 - 2 = 2 \quad \Leftrightarrow \quad x^2 = 4 \quad \Rightarrow \quad x_1 = -2, \quad x_2 = 2,$$

$$x_1 \in I, \quad x_2 \in I, \quad I = I_1 + I_2, \quad I_1 = [-2,\ 2], \quad I_2 = [2,\ 3]$$

$$A = A_1 + A_2 = \int_{-2}^{2} (2 - (x^2 - 2))\, dx + \int_{2}^{3} (x^2 - 2 - 2)\, dx = 2 \int_{0}^{2} (4 - x^2)\, dx +$$

$$+ \int_{2}^{3} (x^2 - 4)\, dx = \frac{32}{3} + \frac{7}{3} = 13$$

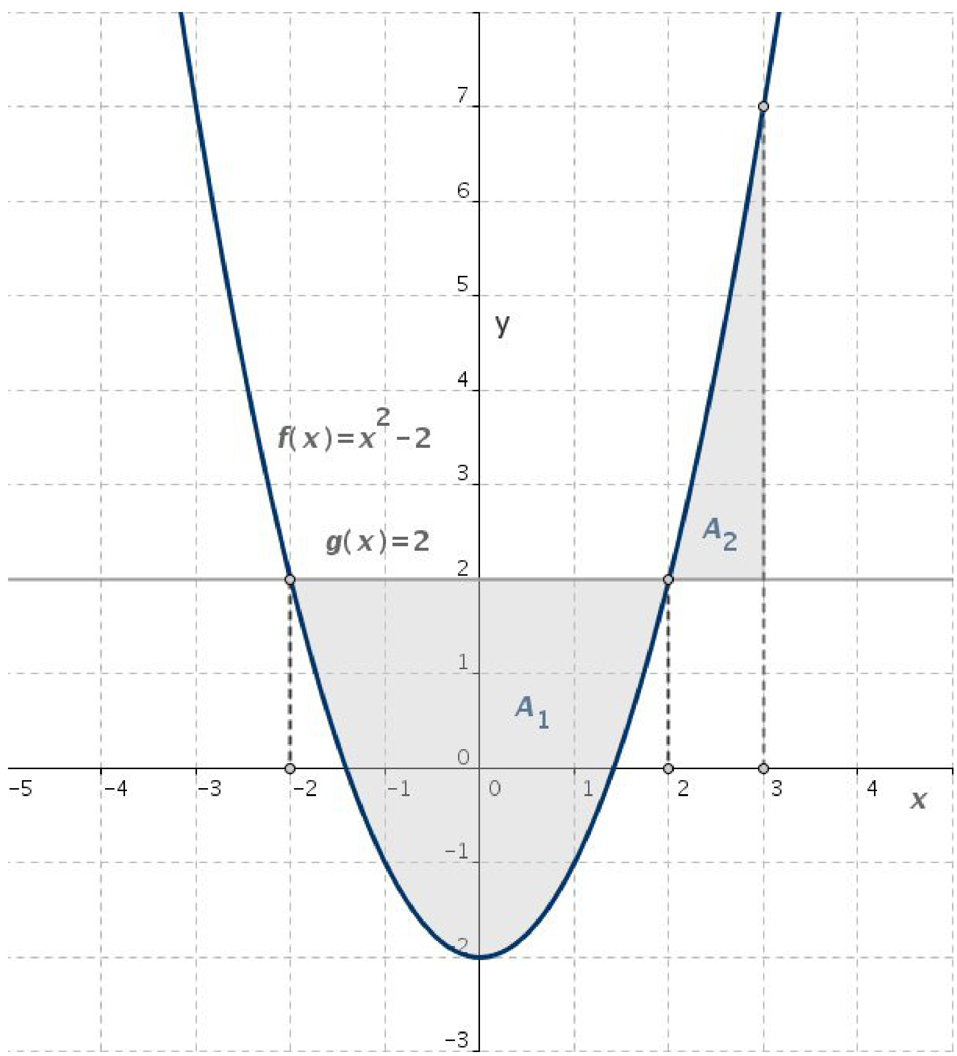

Fig. 4: Die Fläche zwischen der Parabel $f(x) = x^2 - 2$ und der Geraden $g(x) = 2$ im Intervall $I = [-2,\ 3]$

L28

$$\begin{aligned}
f(x) &= x^2 - 2x, \quad g(x) = x, \quad I = [-1, 3] \\
f(x) &= g(x): \quad x^2 - 2x = x \quad \Leftrightarrow \quad x^2 - 3x = x\,(x-3) = 0 \quad \Rightarrow \quad x_1 = 0, \quad x_2 = 3, \\
x_1 &\in I, \quad x_2 \in I, \quad I = I_1 + I_2, \quad I_1 = [-1, 0], \quad I_2 = [0, 3] \\
A &= \int_{-1}^{0} (f(x) - g(x))\,dx + \int_{0}^{3} (g(x) - f(x))\,dx = \int_{-1}^{0} (x^2 - 3x)\,dx + \int_{0}^{3} (3x - x^2)\,dx = \\
&= \frac{11}{6} + \frac{9}{2} = \frac{38}{6} \simeq 6.33
\end{aligned}$$

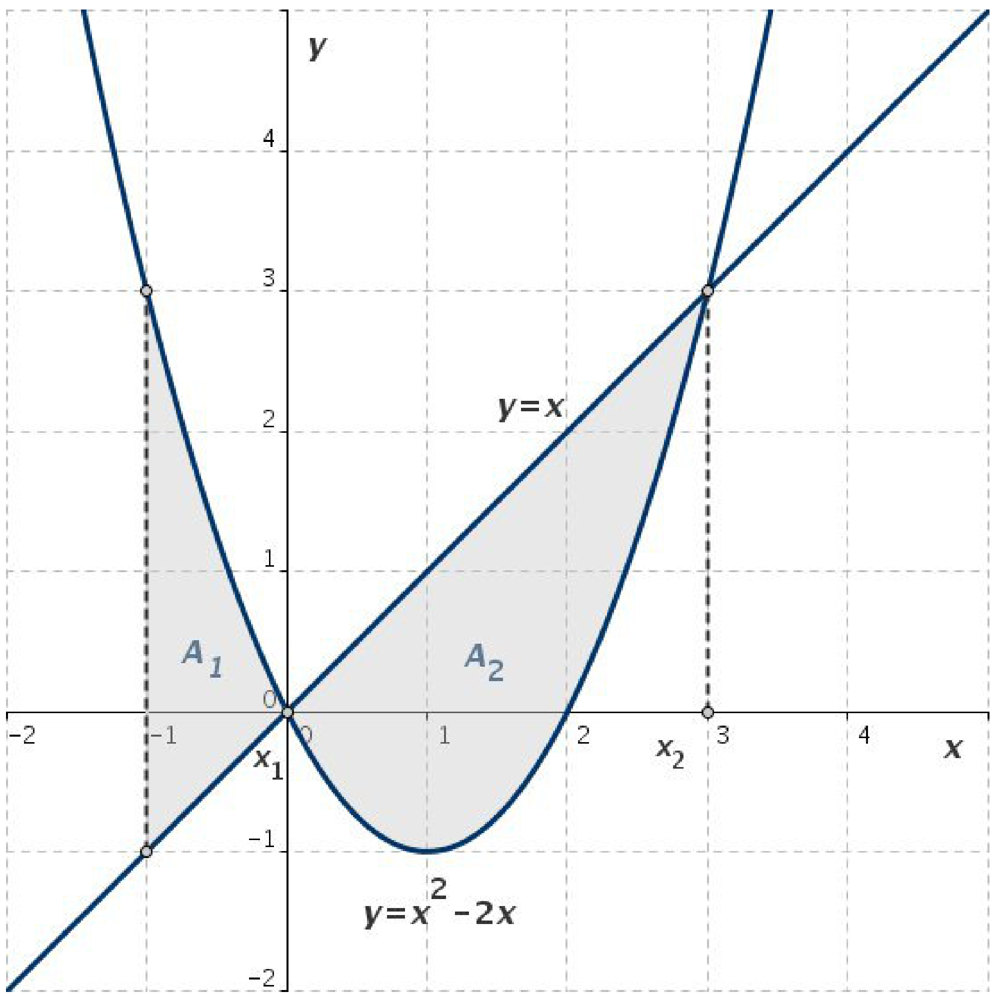

Fig. 5: Die Fläche zwischen der Parabel $f(x) = x^2 - 2x$ und der Geraden $g(x) = x$ im Intervall $[-1, 3]$

L29

$$
\begin{aligned}
f(x) &= x^2 - 2x, \quad g(x) = x + 1, \quad I = [-1,\, 2] \\
f(x) &= g(x): \quad x^2 - 2x = x + 1 \quad \Leftrightarrow \quad x^2 - 3x - 1 = 0 \quad \Rightarrow \quad x_1 = -0.3, \quad x_2 = 3.3, \\
x_1 &\in I, \quad x_2 \notin I, \quad I = I_1 + I_2, \quad I_1 = [-1,\, -0.3], \quad I_2 = [-0.3,\, 2] \\
A &= \int_{-1}^{-0.3} (f(x) - g(x))\, dx + \int_{-0.3}^{2} (g(x) - f(x))\, dx = \int_{-1}^{-0.3} (x^2 - 3x - 1)\, dx + \int_{-0.3}^{2} (1 + 3x - x^2)\, dx = \\
&= 0.99 + 5.49 \;\; (??)
\end{aligned}
$$

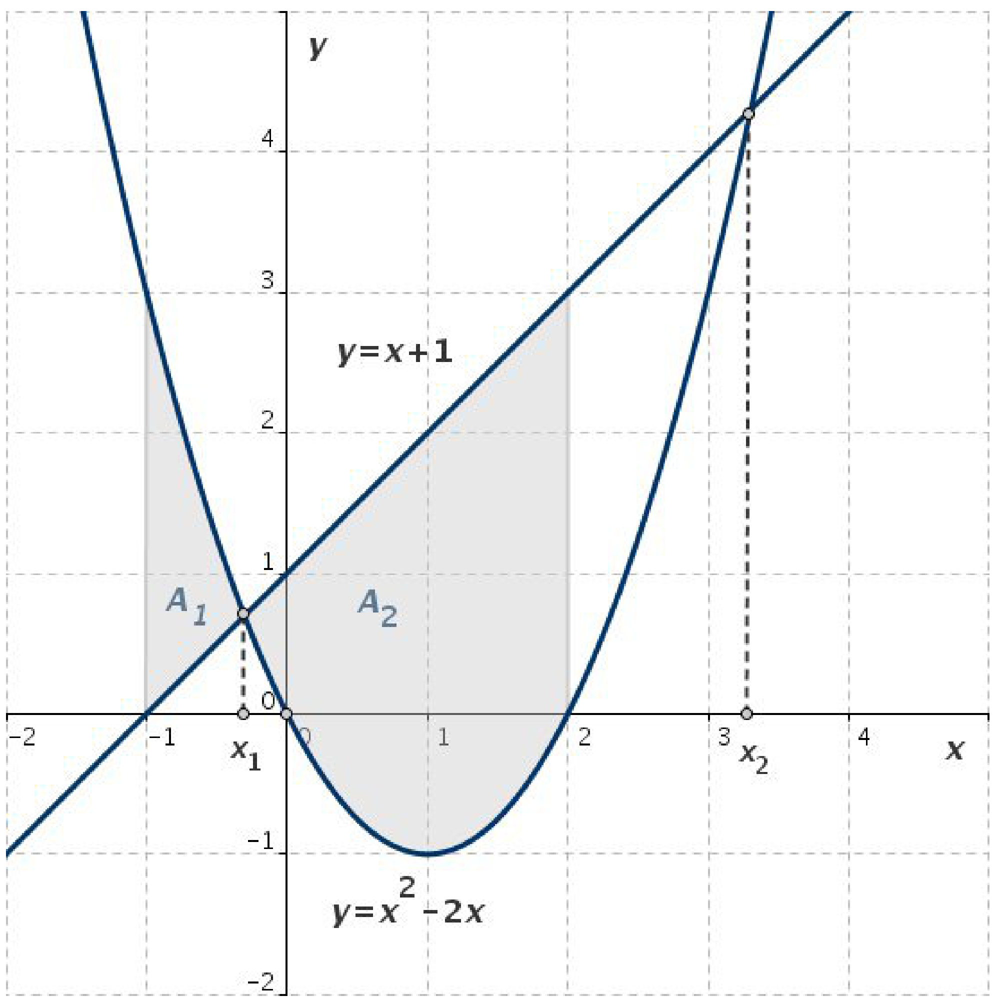

Fig. 6: Die Fläche zwischen der Parabel $f(x) = x^2 - 2x$ und der Geraden $g(x) = x + 1$ im Intervall $[-1,\, 2]$

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