"Every activity worth doing has a learning curve ."

Seth Gordin

7 Curbe

Curbe in natura

Ecuatia unei curbe catenare are froma:

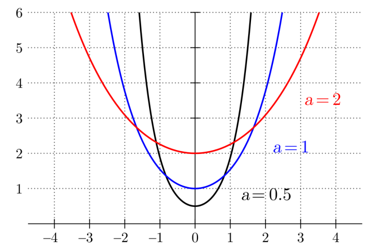
$$y = a\cosh\left(\frac{x}{a}\right) = \frac{a(e^{\frac{x}{a}} + e^{-\frac{x}{a}})}{2}$$

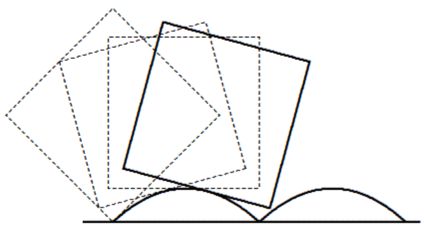
Toate curbele catenare sunt oarecum similare intre ele. Schimbarea parametrului a este echivalenta cu o scalare uniforma a crubei.

Problema rotilor patrate: Care ar trebui sa fie forma drumului pentru ca o roata patrata sa se poata rostogoli fara dificultate ?

Raspuns: O roata patrata se rostogoleste usor pe curbe catenare inversate.

Panza de paianjan este o retea de curbe numite curbe catenare. In fizica sau geometrie, o curba catenara este curba pe care un lant sau cablu il face sub presiunea propriei greutati, atunci cand este prins doar la capete. Curbele de aceasta forma sunt foarte prezente in natura, fiind usor de remarcat.





Sinteza teorie:

- in aceasta fisa ne vom concentra mai mult pe curbele spatiale

Curbe spatiale celebre

- ecuatiile parametrice ale unei curbe spatiale sunt:

$$c: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \quad t \in I \subset \mathbb{R},$$

unde $x(t), y(t), z(t)$ sunt functii in t.

- uneori ecuatiile parametrice sunt date in forma vectoriala:

$$c: \quad \boldsymbol{\bar{r}(t)} = x(t) \cdot \bar{i} + y(t) \cdot \bar{j} + z(t) \cdot \bar{k}$$

unde $\bar{r}(t)$ este vectorul de pozitie al unui punct oarecare $M(t)$ of de pe curba.

- numim o curba $c: I \to \mathbb{R}^3$ regulata daca $\boldsymbol{r'(t)} \neq 0, \quad \forall t \in I$ si vom lucra doar cu astfel de curbe

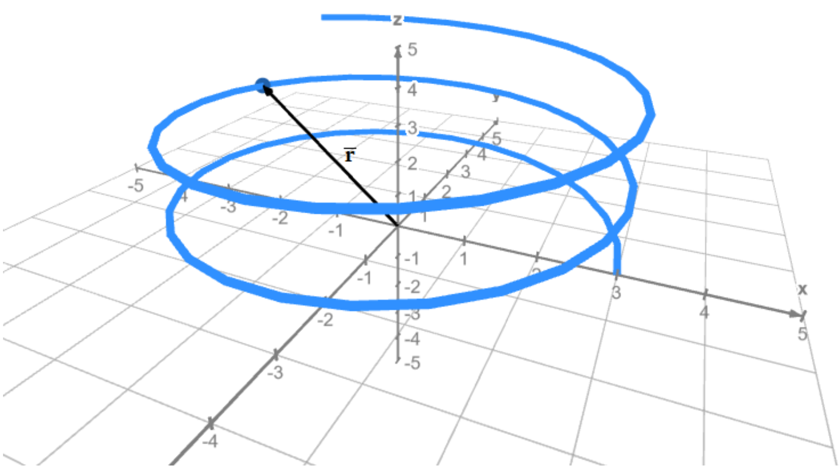
- o curba $3D$ poate fi de asemenea data ca intersectia unor suprafete prin (ecuatiile implicite ale curbei:

$$c: \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$$

unde $F(x, y, z) = 0$ si $G(x, y, z) = 0$ sunt ecuatiile implicite ale suprafetelor.

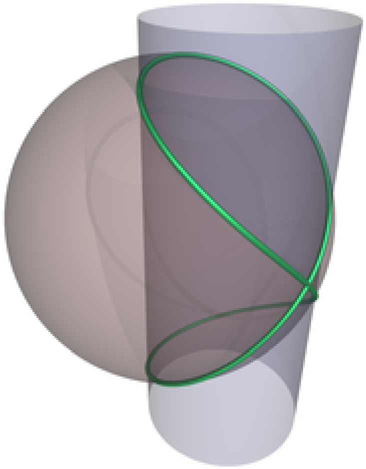
1) Elicea are ecuatiile parametrice

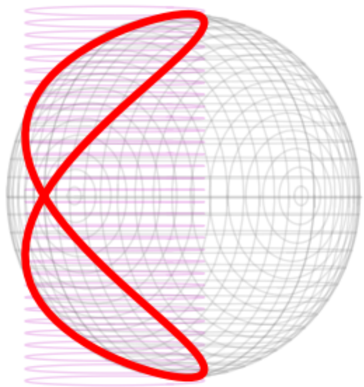
$$c: \begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt \end{cases}, \quad a, b \text{ constante}$$



- are o curbura constanta si o torsiune constanta
- elicea obtinuta pentru $a = 3$ si $b = 1$ este desenata in figura anterioara.

2) Curba lui Viviani poate fi vizualizata ca intersectia dintre un cilindru si o sfera. Arata ca un numar 8 situat pe o sfera:





Daca vom considera cilindrul cu centrul in $(a, 0, 0)$ si de raza a:

$$(x-a)^2 + y^2 = a^2, \qquad (\ F(x,y,z) := (x-a)^2 + y^2 - a^2 \ )$$

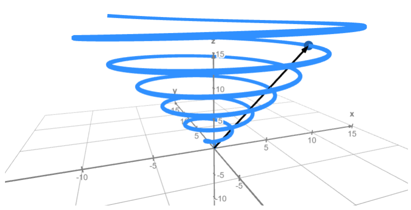
si sfera:

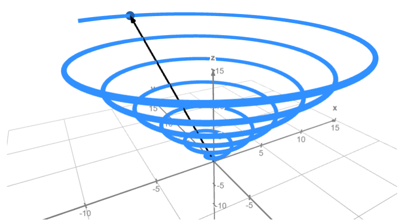
$$x^2 + y^2 + z^2 = 4a^2, \qquad (\ G(x,y,z) := x^2 + y^2 + z^2 - 4a^2 \ )$$

cu centrul in $(0, 0, 0)$ si de raza $2a$, atunci intersectia lor va fi curba Viviani de ecuatii parametrice:

$$c : \begin{cases} x = a(1 + \cos t) \\ y = a \sin t \\ z = 2a \sin\left(\frac{t}{2}\right) \end{cases} \quad , \qquad a \text{ constanta}$$

3) the Elicea conica este o spirala trei dimensionala:

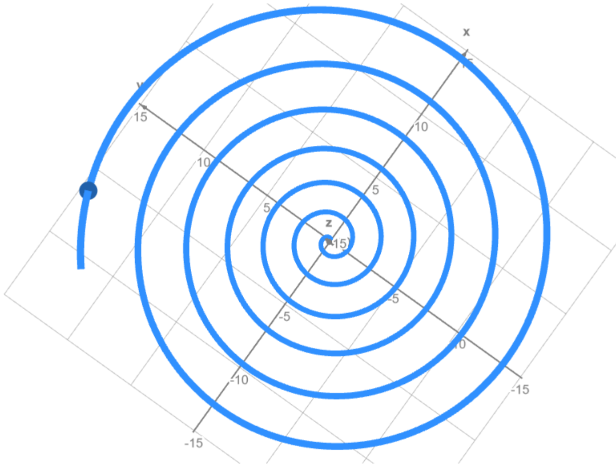




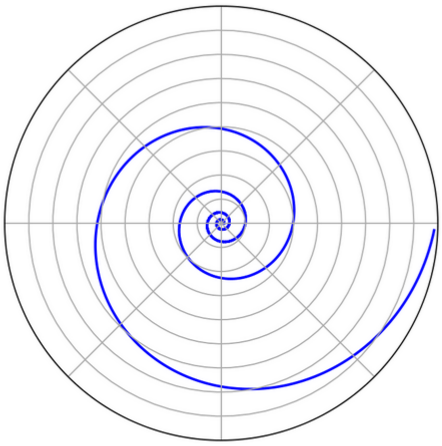
Ecuatiile parametrice posibile sunt de forma:

$$c : \begin{cases} x = t \cos(at) \\ y = t \sin(at) \\ z = bt \end{cases} \quad , \qquad a, b, \text{ constante}$$

O privire de deasupra este data mai jos:



In practica cele mai intersante spirale 3D sunt cele numite spirale logaritmice, care au o proiectie 2D de forma urmatoare:



Mai multe despre spirale puteti gasi aici.

- Ecuatiile parametrice posibile ale unei spirale 3D logaritmice sunt:

$$c: \begin{cases} x = ae^{bt}\cos t \\ y = ae^{bt}\sin t \\ y = ct \end{cases} \quad , \quad a, b, c \text{ constante}$$

Puteti sa generati alte curbe 3D folosind acest link

↳ dupa ce alegeti parametrizarea dati click pe **Redraw Display**

↳ tineti apasat butonul din stanga al mouse-ului pentru a misca graficul si a obtine **diferite perspective ale curbelor generate**

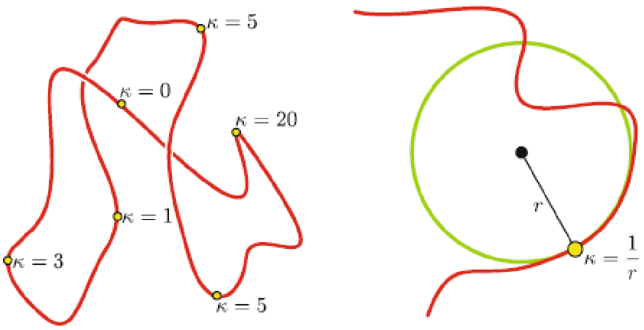
Curbura si torsiunea

- curbura κ este cantitatea prin care curba deviaza de la a fi o curba dreapta

↳ curbura unei drepte este 0 si curbura unui cerc de raza r este constanta in fiecare punct: $\kappa = \frac{1}{r}$

• pentru o curba 2D $c: \quad \bar{r}(t) = x(t) \cdot \bar{i} + y(t) \cdot \bar{j}$ curbura in fiecare punct arbitrar $M(t_0)$ este definita ca:

$$\kappa(t_0) = \frac{|x'(t_0)y''(t_0) - y'(t_0)x''(t_0)|}{\left[(x'(t_0))^2 + (y'(t_0))^2\right]^{\frac{3}{2}}}$$



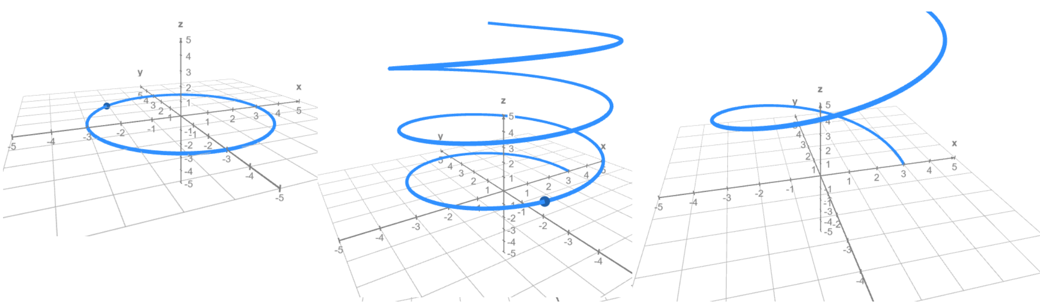
• intr-un punct arbitrar $M(t_0)$ al unei curbe regulate spatiale c curbura este definita prin:

$$\kappa(t_0) = \frac{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\|}{\|\bar{r}'(t_0)\|^3}$$

• torsiunea τ a unei curbe este cantitatea care arata cu cat curba deviaza de la a fi o curba plana.

• intr-un punct oarecare $M(t_0)$ aceasta este definita prin

$$\tau(t_0) = \frac{\left|\left(\bar{r}'(t_0), \bar{r}''(t_0), \bar{r}'''(t_0)\right)\right|}{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\|^2}$$



a) torsiune zero b) torsiune medie c) torsiune mare

Reperul Frenet-Serret

De acum vom considera doar curbe $c : I \to \mathbb{R}^3$ care sunt de clasa C^2 si pentru care $\mathbf{\bar{r}'(t) \times \bar{r}''(t) \neq 0}$, oricare ar fi $t \in I$. Fie asadar $M(t_0)$ un punct arbitrar pe o astfel de curba c.

• elementele reperului Frenet-Serret, numit si reper TNB, sunt

versorul tangentei in M:

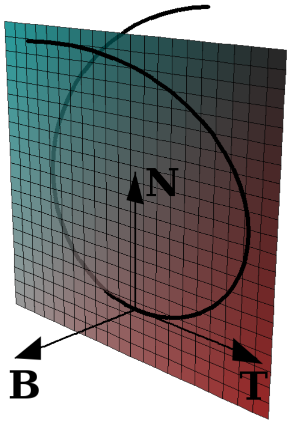
$$\bar{t} = \frac{\bar{r}'(t_0)}{\|\bar{r}'(t_0)\|}$$

versorul binormalei in M:

$$\bar{b} = \frac{\bar{r}'(t_0) \times \bar{r}''(t_0)}{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\|}$$

versorul normalei principale in M:

$$\bar{n} = \bar{b} \times \bar{t} = \frac{\big(\bar{r}'(t_0) \times \bar{r}''(t_0)\big) \times \bar{r}'(t_0)}{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\| \cdot \|\bar{r}'(t_0)\|}$$

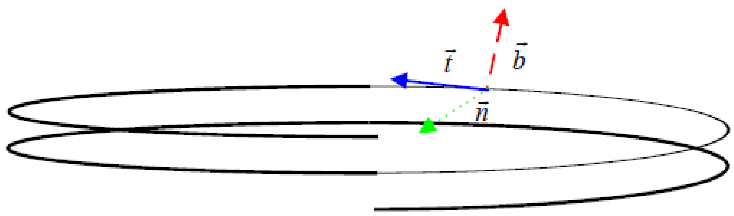


Axele reperului:

• tangenta la c in $M(t_0)$:
↳ are directia data de vectorul $\bar{r}'(t_0)$

• binormala la c in $M(t_0)$:
↳ are directia data de vectorul $\bar{r}'(t_0) \times \bar{r}''(t_0)$

• normala principala la c in $M(t_0)$:
↳ are directia data de vectorul $(\bar{r}'(t_0) \times \bar{r}''(t_0)) \times \bar{r}'(t_0)$



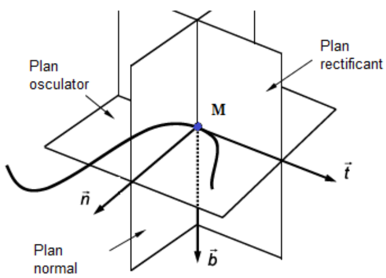
Vizualizare a reperului Frenet-Serret

Planele reperului:

• planul osculator este determinat de M si are ca vector normal pe $\bar{b}(t_0)$

• planul normal este determinat de M si are ca vector normal pe $\bar{t}(t_0)$

• planul rectificant este determinat de M si are ca vector normal pe $\bar{n}(t_0)$



Probleme rezolvate

Problem 1. *Consideram curba date prin ecuatiile parametrice*

$$c: \begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = 3t \end{cases} \quad , \quad t \in [0, 2\pi]$$

i) Gasiti elementele reperului Frenet-Serret in punctul $M(2,0,0)$.

ii) Aflati lungimea segmentului de curba AB, unde $A(0)$ si $B(\pi)$

Solutie: Parametrul corespunzator lui M va fi $t_0 = 0$ deoarece $2\cos 0 = 2$, $2\sin 0 = 0$ si $3 \cdot 0 = 0$. Ecuatiile parametrice vectoriale ale lui c sunt:

$$\bar{r}(t) = 2\cos t\bar{i} + 2\sin t\bar{j} + 3t\bar{k}.$$

Obtinem $\bar{r}'(0) = 2\bar{j} + 3\bar{k}$ si $\bar{r}''(0) = -2\bar{i}$. Versorul tangentei in M va fi:

$$\bar{t}_M = \frac{\bar{r}'(0)}{\|\bar{r}'(0)\|} = \frac{2\bar{j} + 3\bar{k}}{\sqrt{0^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{13}}(0, 2, 3)$$

Pentru versorul binormalei avem nevoie de:

$$\bar{r}'(0) \times \bar{r}''(0) \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 2 & 3 \\ -2 & 0 & 0 \end{vmatrix} = -6\bar{j} + 4\bar{k}$$

Astfel:

$$\bar{b}_M = \frac{-6\bar{j} + 4\bar{k}}{\| -6\bar{j} + 4\bar{k}\|} = \frac{1}{\sqrt{52}}(-6\bar{j} + 4\bar{k})$$

In final versorul normalei principale este:

$$\bar{n}_M = \bar{b}_M \times \bar{t}_M = -\bar{i}$$

In contiuare vom afla ecuatiile dreptelor si planelor apartinand reperului Frenet-Serret in punctul M. Dreapta tangenta care trece prin $M(2,0,0)$ si are directia data de $\bar{r}'(0) = 2\bar{j} + 3\bar{k} = (0,2,3)$ este:

$$\frac{x-2}{0} = \frac{y-0}{2} = \frac{z-0}{3}$$

Binormala care trece prin M si are directia data de vectorul $\bar{r}'(0) \times \bar{r}''(0) = (0, -6, 4)$ este:

$$\frac{x-2}{0} = \frac{y-0}{-6} = \frac{z-0}{4}$$

The principal normal passes through M and has the direction given by the vector $\left(\bar{r}'(0) \times \bar{r}''(0)\right) \times \bar{r}'(0) = -26\bar{i}$:

$$\frac{x-2}{1} = \frac{y-0}{0} = \frac{z-0}{0}$$

Fact: if a vector v gives a direction then $c \cdot v$ gives the same direction. $(c = \frac{1}{-26})$

The osculating plane passes through $M(2,0,0)$ and its normal vector is $\bar{b}_M$ so its equation will be:

$$-6(y-0) + 4(z-0) = 0 \iff -3y + 2z = 0$$

The normal plane passes through $M(2,0,0)$ and its normal vector is $\bar{t}_M$ so its equation will be:

$$2(y-0) + 3(z-0) = 0 \iff 2y + 3z = 0$$

The rectifying plane passes through $M(2,0,0)$ and its normal vector is $\bar{n}_M$ so its equation will be:

$$1(x-2) + 0(y-0) + 0(z-0) = 0 \iff x = 2$$

ii) The length of the chord between two points $M_1(t_1)$ and $M_2(t_2)$ is given by the formula:

$$\ell_{M_1M_2} = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$$

Hence:

$$\ell_{AB} = \int_0^{\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (3)^2}dt = \int_0^{\pi} \sqrt{13}dt = \sqrt{13}\pi.$$

Problem 2. *Find the curvature and the torsion of the curve given by the parametric equations:*

$$c : \begin{cases} x = e^t \\ y = e^{-t} \\ z = t\sqrt{2} \end{cases}, \quad t \in \mathbb{R}$$

Solution: For an arbitrary point $M(t_0) \in c$ the formulae of the curvature and torsion are:

$$\kappa(t_0) = \frac{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\|}{\|\bar{r}'(t_0)\|^3}, \quad \tau(t_0) = \frac{\left|\left(\bar{r}'(t_0), \bar{r}''(t_0), \bar{r}'''(t_0)\right)\right|}{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\|^2}$$

First of all, the **position vector** of a point $M(t)$ is given by:

$$\boldsymbol{\bar{r}(t)} = e^t\bar{i} + e^{-t}\bar{j} + t\sqrt{2}\bar{k}$$

Simple computations lead to:

$$\bar{r}'(t) = e^t\bar{i} - e^{-t}\bar{j} + \sqrt{2}\bar{k}$$

and:

$$\bar{r}''(t) = e^t\bar{i} + e^{-t}\bar{j}, \quad \bar{r}'''(t) = e^t\bar{i} - e^{-t}\bar{j}$$

Thus:

$$\bar{r}'(t) \times \bar{r}''(t) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ e^t & -e^{-t} & \sqrt{2} \\ e^t & -e^{-t} & 0 \end{vmatrix} = -e^{-t}\sqrt{2}\bar{i} + e^t\sqrt{2}\bar{j} + 2\bar{k}$$

The necessary norms are:

$$\|\bar{r}'(t)\| = \sqrt{e^{2t} + e^{-2t} + 2} = e^t + e^{-t}, \quad \|\bar{r}'(t) \times \bar{r}''(t)\| = \sqrt{2e^{-2t} + 2e^{2t} + 4} = \sqrt{2}(e^{-t} + e^t)$$

Thus the curvature at $M(t_0)$ will be:

$$\kappa(t_0) = \frac{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\|}{\|\bar{r}'(t_0)\|^3} = \frac{\sqrt{2}(e^{-t_0} + e^{t_0})}{(e^{t_0} + e^{-t_0})^3} = \sqrt{2}(e^{t_0} + e^{-t_0})^{-2}$$

In order to compute the torsion one needs the triple product:

$$\left(\bar{r}'(t), \bar{r}''(t), \bar{r}'''(t)\right) = \begin{vmatrix} e^t & -e^{-t} & \sqrt{2} \\ e^t & e^{-t} & 0 \\ e^t & -e^{-t} & 0 \end{vmatrix} = -2\sqrt{2}$$

and the torsion at $M(t_0)$ will be:

$$\tau(t_0) = \frac{\left|\left(\bar{r}'(t_0), \bar{r}''(t_0), \bar{r}'''(t_0)\right)\right|}{\|\bar{r}'(t_0) \times \bar{r}''(t_0)\|^2} = \frac{|-2\sqrt{2}|}{2(e^{-t_0} + e^{t_0})^2} = -\kappa(t_0)$$

Problem 3. *Let us consider the curve:*

$$c : \begin{cases} x = 3\sin^2 t \\ y = 3\sin(2t) \\ z = 3\cos^2 t \end{cases}, \quad t \in \mathbb{R}$$

Show that c is a plane curve.

Solution: The torsion τ measures the amount to which c deviates from being a plane curve. In order to be a plane curve one has to have zero torsion at every arbitrary point $M(t_0)$ of this curve. Having in mind the formula of $\tau(t_0)$ it is enough to prove:

$$\left(\bar{r}'(t_0), \bar{r}''(t_0), \bar{r}'''(t_0)\right) = 0, \quad \forall\ t_0 \in \mathbb{R}$$

Straightforward one gets:

$$\bar{r}(t_0) = 3\sin^2 t\bar{i} + 3\sin(2t)\bar{j} + 3\cos^2 t\bar{k}$$

$$\bar{r}'(t_0) = 3\sin(2t)\bar{i} + 6\cos(2t)\bar{j} - 3\sin(2t)\bar{k}$$

$$\bar{r}''(t_0) = 6\cos(2t)\bar{i} - 12\sin(2t)\bar{j} - 6\cos(2t)\bar{k}$$

$$\bar{r}'''(t_0) = -12\sin(2t)\bar{i} - 24\cos(2t)\bar{j} + 12\sin(2t)\bar{k}$$

The triple product will be:

$$(\bar{r}'(t_0), \bar{r}''(t_0), \bar{r}'''(t_0)) = \begin{vmatrix} 3\sin(2t_0) & 6\cos(2t_0) & -3\sin(2t_0) \\ 6\cos(2t_0) & -12\sin(2t_0) & -6\cos(2t_0) \\ -12\sin(2t_0) & -24\cos(2t_0) & 12\sin(2t_0) \end{vmatrix} = 0, \quad \forall t_0 \in \mathbb{R}$$

because the first and the last row are linearly dependent. Finally the curve c will be a plane curve.

Problem 4. *Find the points lying on the curve:*

$$c: \begin{cases} x = 2t - 1 \\ y = t^3 \\ z = 1 - t^2 \end{cases}, \quad t \in \mathbb{R}$$

where the osculating plane is perpendicular to the plane:

$$\alpha: \; 7x - 12y + 5z = 0.$$

Solution: Let us suppose that $M(t_0) \in c$ is a point with the above property. We will try to get some restrictions on t_0 (equations) in order to find all the possible values of t_0. Two planes are perpendicular iff their normal directions are perpendicular.

The normal direction to the osculating plane is:

$$\bar{v} = \bar{r}'(t_0) \times \bar{r}''(t_0) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3t_0^2 & -2t_0 \\ 0 & 6t_0 & -2 \end{vmatrix} = 6t_0^2\bar{i} + 4\bar{j} + 12t_0\bar{k}$$

because $\bar{b}(t_0)$ provides this direction. The normal direction to the plane α is $\bar{n} = (7, -12, 5)$. Thus the necessary condition $\bar{v} \perp \bar{n}$ becomes $\langle v, \bar{n} \rangle = 0 \implies 42t_0^2 - 48 + 60t_0 = 0$ with the roots $t_1 = -2$ and $t_2 = \frac{4}{7}$. As a consequence we get two points $M_1(t_1) = M_1(-5, -8, -3)$ and $M_2(t_2) = M_2(\frac{1}{7}, \frac{64}{343}, \frac{33}{49})$

Problem 5. *Gasiti versorii corespunzatori reperului Frenet-Serret in punctul $M(0,0,0)$ al curbei c data prin:*

$$c: \begin{cases} x = y^2 + z^2 \\ x + y + z = 0 \end{cases}$$

Solutie: Deoarece curba este data ca intersectie de doua suprafete am dori sa aflam ecuatiile parametrice ale lui c. Sa notam pentru inceput

$$F(x, y, z) = x - y^2 - z^2 \qquad (1)$$

si

$$G(x, y, z) = x + y + z \qquad (2).$$

Putem aplica teorema functiilor implicite deoarece:

$$\frac{D(F,G)}{D(z,x)}\bigg|_{(0,0,0)} = \begin{vmatrix} \frac{\partial F}{\partial z} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial z} & \frac{\partial G}{\partial x} \end{vmatrix}\Bigg|_{(0,0,0)} = \begin{vmatrix} -2z & 1 \\ -1 & 1 \end{vmatrix}\Bigg|_{(0,0,0)} = -1 \neq 0$$

Astfel intr-o vecinatate a lui $(0,0,0) \in \mathbb{R}^3$ exista functiile $z(y)$ si $x(y)$ cu proprietatea $z = z(y)$ si $x = x(y)$. In aceasta vecinatate ecuatia parametrica va fi

$$\bar{\mathbf{r}}(y) = x(y)\bar{i} + y\bar{j} + z(y)\bar{k}$$

Alegand $y = t$ obtinem forma obisnuita:

$$\bar{\mathbf{r}}(t) = x(t)\bar{i} + t\bar{j} + z(t)\bar{k}$$

In aceasta parametrizare punctul M corespunde unui parametru t_0. Daca folosim faptul ca $\bar{r}(t_0) = (0,0,0)$ observam ca M corespunde lui $t_0 = 0$.

In continuare putem inlocui $y = t,\ x = x(t), z = z(t)$ in (1) si (2) pentru a obtine:

$$x(t) - t^2 - z(t)^2 = 0,$$
$$x(t) + t + z(t) = 0.$$

Pentru a determina versorii reperului Frenet-Serret in M trebuie sa gasim vectorii $\bar{r}'(0)$ and $\bar{r}''(0)$. Ideea este ca putem diferentia relatiile de mai sus pentru a afla $x'(0), x''(0), z'(0)$ si $z''(0)$.

Pentru inceput se obtine usor:

$$x'(t) - 2t - 2z(t) \cdot z'(t) = 0$$

si

$$x'(t) + 1 + z'(t) = 0$$

Inlocuind $t_0 = 0$, $x(0) = 0, z(0) = 0$ obtinem $x'(0) = 0$ si $z'(0) = -1$, prin urmare:

$$\bar{r}'(0) = 0 \cdot i + 0 \cdot j - 1 \cdot k.$$

Daca derivam inca o data obtinem:

$$x''(t) - 2 - 2z(t) \cdot z''(t) - 2[z''(t)]^2 = 0$$

si, respectiv

$$x''(t) + z''(t) = 0$$

Putem folosi informatiile obtinute anterior, $x'(0) = 0$ si $z'(0) = -1$, pentru a obtine $x''(0) = 2$ si $z''(0) = -3$. Astfel:

$$\bar{r}''(0) = 2 \cdot i + 0 \cdot j - 3 \cdot k.$$

In final se folosesc formulele

$$\bar{t} = \frac{\bar{r}'(0)}{\|\bar{r}'(0)\|}$$

$$\bar{b} = \frac{\bar{r}'(0) \times \bar{r}''(0)}{\|\bar{r}'(0) \times \bar{r}''(0)\|}$$

si

$$\bar{n} = \bar{b} \times \bar{t} = \frac{\big(\bar{r}'(0) \times \bar{r}''(0)\big) \times \bar{r}'(0)}{\|\bar{r}'(0) \times \bar{r}''(0)\| \cdot \|\bar{r}'(0)\|}.$$

Probleme propuse

Problem 1. *Aflati punctele situate pe curba:*

$$c: \begin{cases} x = \frac{2}{t} \\ y = \ln t \\ z = -t^2 \end{cases}$$

in care tangenta este paralela cu planul $\alpha: \ x - y + 8z - 1 = 0.$

Problem 2. *Scrieti ecuatiile versorilor, axelor si planelor reperului Frenet-Serret corespunzator curbei:*

$$c: \quad \bar{r}(t) = 2t \cdot \bar{i} + t^2 \cdot \bar{j} + \ln t \cdot \bar{k}, \qquad t \in (0, \infty)$$

in punctul $M(t_0 = 1)$.

Problem 3. *Aflati ecuatiile binormalelor corespunzatoare:*

$$c: \quad x^2 - y^2 = z, \ 2x = 3y^2$$

in punctele M *in care acestea sunt paralele cu planul* yOz.

Hint: noteaza $y = t$ si obtine ecuatia parametrica a lui c.

Problem 4. *Calculati curbura intr-un punct arbitrar al curbei:*

$$c: \quad \begin{cases} x = a(t + \sin t) \\ y = a(1 - \cos t) \end{cases} \quad , \quad t \in [0, 2\pi]$$

Problem 5. *Gasiti elementele reperului Frenet-Serret frame in punctul* $M(1, -1, 2)$ *al curbei:*

$$c: \quad \begin{cases} z = x^2 + y^2 \\ x + y + z = 2 \end{cases}$$

Problem 6. *Aratati ca urmatoarea curba:*

$$c: \quad \begin{cases} x = 3 + 2t + 4t^3 \\ y = 4 + 3t + 2t^3 \\ z = 2 + 4t + 3t^3 \end{cases}$$

este o curba plana si aflati ecuatia planului in care se situeaza.

Problem 7. *Sa se gaseasca vectorii de pozitie ale punctelor* M *de pe curba:*

$$c: \quad x = \ln t, \ y = t, \ z = -\frac{1}{t}$$

unde binormala este paralela cu planul $\alpha: \ x + 3y - z = 0.$

Problem 8. *Sa se calculeze curbura si torsiunea curbelor:*

i) $x = e^t, \ y = e^{-t}, \ z = \sqrt{2}t$ *in* $M(1, 1, 0)$

ii) $x^2 + y^2 = 8, \ y^2 + z^2 = 8$ *in* $M(1, 1, 1)$