"Creativitatea se naste din abilitatea de a vedea lucrurile din cat mai multe unghiuri."

Keri Smith

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Geometria spatiilor euclidiene $\mathbb{R}^n$

Sinteza teorie

• vectorul $\overrightarrow{AB}$ dat de segmentul orientat $[AB]$ are in plan coordonatele date de $(x_B - x_A, y_B - y_A)$ si in spatiu $(x_B - x_A, y_B - y_A, z_B - z_A)$

↳ formula se generalizeaza usor pentru $\mathbb{R}^n$

• produsul scalar al vectorilor $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ si $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ este:

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

• unghiul θ dintre vectorii $\mathbf{u}$ si $\mathbf{v}$ este dat de formula:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}$$

unde $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$ este lungimea vectorului

Vectorii $\mathbf{u}$ si $\mathbf{v}$ sunt ortogonali (perpendiculari) $\Longleftrightarrow \langle \mathbf{u}, \mathbf{v} \rangle = 0$

• proiectia ortogonala a unui vector $\mathbf{v}$ pe un vector $\mathbf{u}$ este data prin:

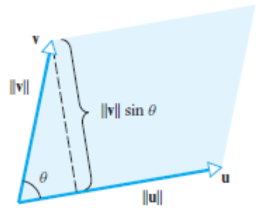
$$\mathrm{proj}_{\mathbf{u}}\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \cdot \mathbf{u}$$

• produsul vectorial a doi vectori $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ este:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Aplicatie: aria paralelogramului determinat de catre vectorii $\mathbf{u}$ si $\mathbf{v}$ este data de lungimea vectorului $\mathbf{u} \times \mathbf{v}$, adica $\|\mathbf{u} \times \mathbf{v}\|$, si are loc formula:

$$\begin{aligned}\mathcal{A}_{paralelogram} &= \|\mathbf{u} \times \mathbf{v}\| \\ &= \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot \sin\theta\end{aligned}$$



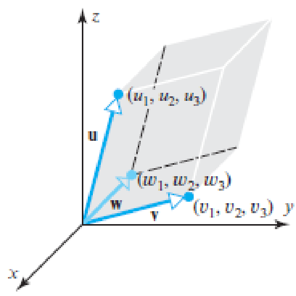
unde θ este unghiul dintre acestia

- produsul mixt a trei vectori $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ si $\mathbf{w} = (w_1, w_2, w_3)$ este dat de formula:

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

si modulul acestuia este volumul paralelipipedului cu muchiile date de catre vectorii $\mathbf{u}, \mathbf{v}, \mathbf{w}$:

$$V_{paralelipiped} = |(\mathbf{u}, \mathbf{v}, \mathbf{w})|$$



Vectorii $\mathbf{u}, \mathbf{v}, \mathbf{w}$ sunt coplanari $\Longleftrightarrow (\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0$

- **Proprietati elementare ale celor trei produse:**

i) distanta dintre doi vectori se calculeaza cu formula:

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

ii) are loc formula $\|\alpha\mathbf{u}\| = |\alpha| \cdot \|\mathbf{u}\|$ pentru norma unui vector scalat cu $\alpha \in \mathbb{R}$
iii) $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$ produsul scalar liniar in ambele argumente
iv) $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$
v) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
vi) $\mathbf{u} \times (\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha \cdot \mathbf{u} \times \mathbf{v} + \beta \cdot \mathbf{u} \times \mathbf{w}$ produsul vectorial este liniar in ambele argumente
vii) $(\alpha\mathbf{u} + \beta\mathbf{v}) \times \mathbf{w} = \alpha \cdot \mathbf{u} \times \mathbf{w} + \beta \cdot \mathbf{v} \times \mathbf{w}$
viii) $\mathbf{u} \times \mathbf{u} = \bar{0}$
ix) $(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle$ legatura dintre cele 3 produse

Probleme rezolvate

Problema 1. *Se dau vectorii* $\mathbf{v} = \mathbf{a} + 3\mathbf{b}$ *si* $\mathbf{u} = \mathbf{a} - 2\mathbf{b}$ *cu proprietatile* $\|\mathbf{a}\| = 6$ *si* $\|\mathbf{b}\| = 4$ *si* $\theta = \widehat{(\mathbf{a}, \mathbf{b})} = \frac{\pi}{6}$. *Sa se calculeze produsul scalar al vectorilor* $\mathbf{u}$ *si* $\mathbf{v}$ *precum si aria paralelogramului construit pe vectorii* $\mathbf{u}$ *si* $\mathbf{v}$.

Solutie: Incepem prin a calcula produsul scalar. Vom folosi proprietatile produsului scalar

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= \langle \mathbf{a} - 2\mathbf{b}, \mathbf{a} + 3\mathbf{b} \rangle = \langle \mathbf{a} - 2\mathbf{b}, \mathbf{a} \rangle + \langle \mathbf{a} - 2\mathbf{b}, 3\mathbf{b} \rangle \\ &= \langle \mathbf{a} - 2\mathbf{b}, \mathbf{a} \rangle + 3\langle \mathbf{a} - 2\mathbf{b}, \mathbf{b} \rangle \\ &= \langle \mathbf{a}, \mathbf{a} \rangle - 2\langle \mathbf{b}, \mathbf{a} \rangle + 3\langle \mathbf{a}, \mathbf{b} \rangle - 6\langle \mathbf{b}, \mathbf{b} \rangle \\ &= \|\mathbf{a}\|^2 + \langle \mathbf{a}, \mathbf{b} \rangle - 6\|\mathbf{b}\|^2\end{aligned}$$

Folosim acum formula:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos(\widehat{\mathbf{a}, \mathbf{b}}) = 6 \cdot 4 \cdot \cos\frac{\pi}{6} = 24 \cdot \frac{\sqrt{3}}{2} = 12\sqrt{3}$$

In concluzie:

$$\langle \mathbf{u}, \mathbf{v} \rangle = 6^2 + 12\sqrt{3} - 6 \cdot 4^2 = 12\sqrt{3} - 60$$

Pentru aria paralelogramului amintim formula:

$$Aria_{paralelogram} = \|\mathbf{u} \times \mathbf{v}\|$$

Vom calcula intai produsul vectorial folosind proprietatile acestuia:

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (\mathbf{a} + 3\mathbf{b}) \times (\mathbf{a} - 2\mathbf{b}) = \mathbf{a} \times \mathbf{a} + 3\mathbf{b} \times \mathbf{a} - 2\mathbf{a} \times \mathbf{b} + 6\mathbf{b} \times \mathbf{b} \\ &= \bar{0} + 3\mathbf{b} \times \mathbf{a} - 2\mathbf{a} \times \mathbf{b} + \bar{0} \qquad (\text{caci } \mathbf{a} \times \mathbf{a} = \bar{0}) \\ &= 5\mathbf{b} \times \mathbf{a} \qquad (\text{caci } \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a})\end{aligned}$$

Asadar:

$$Aria_{paralelogram} = \|5\mathbf{b} \times \mathbf{a}\| = 5\|\mathbf{b}\| \cdot \|\mathbf{a}\| \cdot \sin(\widehat{\mathbf{b}, \mathbf{a}}) = 5 \cdot 4 \cdot 6 \cdot \sin\frac{\pi}{6} = 60$$

Problema 2. *Se dau punctele* $A(2, 2, 1)$ *si* $B(4, 1, 3)$. *Se cere:*

i) Lungimea vectorului $\overline{AB}$

ii) Sa se determine un vector $\mathbf{v}$ *in planul* xOy *astfel incat:*

$$\mathbf{v} \perp \overline{AB} \quad si \quad \|\mathbf{v}\| = \|\overline{AB}\|$$

Solutie: Vectorul $\overline{AB}$ are componentele:

$$\overline{AB} = (4 - 2, 1 - 2, 3 - 1) = (2, -1, 2)$$

deci
$$\|\overline{AB}\| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$$
Cautam acum un vector $\mathbf{v}$ 3-dimensional aflat in planul xOy, deci va avea coordonata z nula:
$$v = (x, y, 0).$$
Doarece $\mathbf{v} \perp \overline{AB}$ obtinem:
$$\langle \mathbf{v}, \overline{AB} \rangle = 0$$
adica:
$$\langle (x, y, 0), (2, -1, 2) = 2 \cdot x - 1 \cdot y + 2 \cdot 0 = 0$$
si astfel obtinem informatia $2x = y$. Cautam acum sa extragem informatii si din faptul ca $\|\mathbf{v}\| = \|\overline{AB}\|$:
$$\|v\| = \sqrt{x^2 + y^2 + 0^2} = \|\overline{AB}\| = 3$$
Deci $x^2 + y^2 = 9$. Folosind cele doua informatii obtinute din prelucrarea datelor avem $x^2 + (2x)^2 = 9$ si prin urmare $x = \pm\sqrt{\frac{9}{5}} = \pm\frac{3}{\sqrt{5}}$. Apoi $y = \pm\frac{6}{\sqrt{5}}$. Deci am obtinut doi vectori $\mathbf{v}$ cu proprietatile date:
$$\mathbf{v} = \left(\frac{3}{\sqrt{5}}, \frac{6}{\sqrt{5}}\right) \quad \text{si} \quad \mathbf{v} = \left(-\frac{3}{\sqrt{5}}, -\frac{6}{\sqrt{5}}\right)$$

Problema 3. *Fie punctele $A(1, 2, -1)$, $B(1, 0, 3)$, $C(2, 1, 2)$ si $D(2, 3, 4)$.*

i) Sa se arate ca A, B, C, D formeaza un tetraedru

ii) Aflati volumul tetraedrului $ABCD$

iii) Aflati proiectia vectorului $\overline{AB}$ pe vectorul $\overline{BC}$

iv) Aflati lungimea inaltimii din A dusa in triunghiul ΔABC.

Soluţie: i) Punctele A, B, C, D formeaza un tetraedru daca nu sunt coplanare, adica daca de exemplu vectorii $\overline{AB}, \overline{AC}$ si $\overline{AD}$ nu sunt coplanari. Mai intai aflam coordonatele acestor vectori:
$$\overline{AB} = (1 - 1, 0 - 2, 3 - (-1)) = (0, -2, 4)$$
$$\overline{AC} = (2 - 1, 1 - 2, 2 - (-1)) = (1, -1, 3)$$
$$\overline{AD} = (2 - 1, 3 - 2, 4 - (-1)) = (1, 1, 5)$$
Pentru a testa coplanaritatea acestora avem nevoie de produsul mixt:
$$(\overline{AB}, \overline{AC}, \overline{AD}) = \begin{vmatrix} 0 & -2 & 4 \\ 1 & -1 & 3 \\ 1 & 1 & 5 \end{vmatrix} = 12 \neq 0$$
prin urmare nu sunt coplanari, asadar A, B, C, D formeaza un tetraedru.

ii) Volumul tetraedrului $ABCD$ este prin definitie modulul produsului mixt al vectorilor care il genereaza, adica $\overline{AB}, \overline{AC}, \overline{AD}$:

$$V_{ABCD} = |(\overline{AB}, \overline{AC}, \overline{AD})| = |12| = 12 \text{ (unitati de volum)}$$

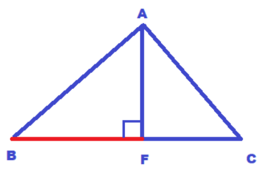
iii) Proiectia ceruta se obtine prin formula:

$$proj_{\overline{BC}} \overline{AB} = \frac{\langle \overline{AB}, \overline{BC} \rangle}{\langle \overline{BC}, \overline{BC} \rangle} \cdot \overline{BC}$$

Coordonatele lui $\overline{BC}$ sunt $\overline{BC} = (1, 1, -1)$ deci:

$$proj_{\overline{BC}} \overline{AB} = \frac{\langle (0, -2, 4), (1, 1, -1) \rangle}{\langle (1, 1, -1), (1, 1, -1) \rangle} \cdot (1, 1, -1) = \frac{-6}{3}(1, 1, -1) = (-2, -2, 2)$$

iv) In triunghiul ΔABC aflam inaltimea AF folosind proiectia lui AB pe BC, care este prin definitie BF:



Aplicand teorema lui Pitagora in triunghiul ΔABF :

$$\|\overline{AB}\|^2 = \|\overline{AF}\|^2 + \|\overline{BF}\|^2$$

dar $\overline{BF} = proj_{\overline{BC}} \overline{AB} = (-2, -2, 2)$ si astfel:

$$(-2)^2 + 4^2 = \|\overline{AF}\|^2 + (-2)^2 + 2^2 + 2^2$$

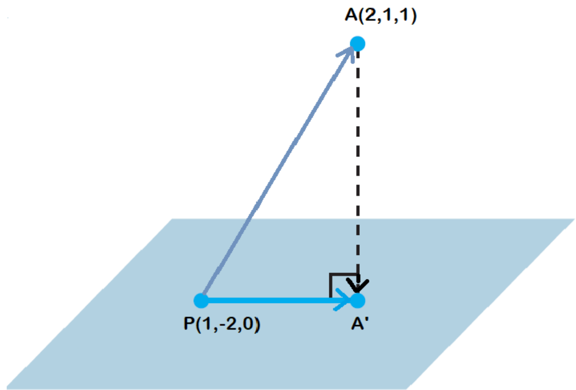
In final obtinem $\|\overline{AF}\| = \sqrt{8} = 2\sqrt{2}$.

Problema 4. *Aflati proiectia punctului* $A(2, 1, 1)$ *pe planul:*

$$\pi \; : \qquad x - 2y - 2z = 5.$$

Solutie: Problema poate fi rezolvata in multiple moduri dar aici dorim sa evidentiam rolul proiectiilor ortogonale si vom da o solutie vectoriala.

Sa consideram pentru inceput un punct P in planul dat. Putem alege $P(1, -2, 0)$ caci coordonatele sale satisfac ecuatia planului. Notam cu A' proiectia lui A pe planul dat.



Putem afla coordonatele lui A' daca aflam coordonatele vectorului $\overline{AA'}$. Reamintim ca:

$$\overline{AA'} = (x_{A'} - x_A, y_{A'} - y_A, z_{A'} - z_A)$$

Pentru a utiliza aceasta informatie trebuie sa mentionam ca exista o singura directie perpendiculara pe un plan si aceasta se numeste directia normala. Directia normala poate fi recuperata din ecuatia unui plan $ax + by + cz + d = 0$ fiind "codata" in coeficientii variabilelor x,y,z. Asadar $\bar{n} = (a, b, c)$ este vectorul care defineste directia normala la plan.

In cazul nostru $\bar{n} = (1, -2, -2)$. Se observa pe desen ca:

$$\overline{AA'} = proj_{\bar{n}} \overline{AP}.$$

Deoarece $\overline{AP} = (-1, -3, -1)$ obtinem din formula proiectiei ortogonale a vectorului $\overline{AP}$ pe vectorul $\bar{n}$:

$$proj_{\bar{n}} \overline{AP} = \frac{\langle(-1,-3,-1),(1,-2,-2)\rangle}{\langle(1,-2,-2),(1,-2,-2)\rangle}(1,-2,-2) = \left(\frac{7}{9}, -\frac{14}{9}, -\frac{14}{9}\right)$$

$$\Longrightarrow \quad \begin{cases} x_{A'} = x_A + \frac{7}{9} = 2 + \frac{7}{9} = \frac{25}{9} \\ y_{A'} = y_A - \frac{14}{9} = 1 - \frac{14}{9} = -\frac{5}{9} \\ z_{A'} = z_A - \frac{14}{9} = 1 - \frac{14}{9} = -\frac{5}{9} \end{cases}$$

In concluzie $A'\left(\frac{25}{9}, -\frac{5}{9}, -\frac{5}{9}\right)$.

Probleme propuse

A. Consolidare cunostinte

Problema A.1. *a) Demonstrati inegalitatea Minkowski:*

$$\sqrt{(a_1+b_1)^2 + (a_2+b_2)^2 + (a_3+b_3)^2} \le \sqrt{a_1^2 + a_2^2 + a_3^2} + \sqrt{b_1^2 + b_2^2 + b_3^2},$$

folosind argumente geometrice.

b) Demonstrati inegalitea Cauchy-Buniakovski-Schwarz:

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

folosind argumente geometrice.

Problema A.2. *Fie* $\mathbf{u} = (4, 1, 1)$, $\mathbf{v} = (0, 4, -2)$ *si* $\mathbf{w} = (1, -2, 3)$
Verificati prin calcul urmatoarele proprietati:

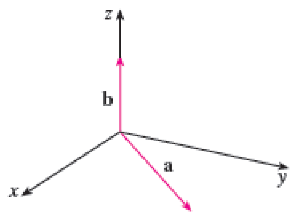
i) $\mathbf{u} \times \mathbf{u} = \bar{0}$

ii) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

iii) $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = -\mathbf{v} \bullet (\mathbf{u} \times \mathbf{w})$

iv $\mathbf{u} \times 2\mathbf{w} = 2(\mathbf{u} \times \mathbf{w})$

Problema A.3. *Figura de mai jos arata un vector* **a** *in planul* xy *si un vector* **b** *in directia axei* Oz. *Lungimile celor doi vectori sunt* $\|\mathbf{a}\| = 3$ *si* $\|\mathbf{b}\| = 2$. *Aflati norma* $\|\mathbf{a} \times \mathbf{b}\|$. *Justificati ce semn au componentele vectorului* $\mathbf{a} \times \mathbf{b}$.



B. Tehnica de calcul

Problema B.1. *Aratati ca vectorii* u *si* v *sunt ortogonali daca si numai daca are loc teorema lui Pitagora:*

$$\|u\|^2 + \|v\|^2 = \|u + v\|^2, \qquad u, v \in \mathbb{R}^n.$$

Problema B.2. *Sa se arate ca vectorii* $u = 2\bar{i} - 6\bar{j}$, $v = \bar{i} + 7\bar{j}$ *si* $w = -3\bar{i} - \bar{j}$ *formeaza laturile unui triunghi si sa se determine unghiurile acestui triunghi.*

Problema B.3. *Sa se determine inaltimea triunghiului* ABC *si aria acestuia daca se stiu:*

i) varfurile sale $A(1, 1, 2), B(2, 2, 1)$ *si* $C(1, 2, 1)$

ii) $\overrightarrow{AB} = \bar{i} + \bar{j} - \bar{k}$ *si* $\overrightarrow{AC} = 2\bar{i} - \bar{j} + 3\bar{k}$

Problema B.4. *Aflati un vector care este perpendicular pe planul ce contine punctele* $A(1,4,6)$, $B(-2,5,-1)$ *si* $C(1,-1,1)$.

Problema B.5. *Aratati ca:*

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \textit{si} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Aratati apoi ca:

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \textit{si} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

Folosind aceste reguli calculati produsul vectorial dintre vectorii $\mathbf{u} = (1,-2,4)$ *si* $\mathbf{v} = (2,1,-3)$.

C. Probleme cu caracter practic-aplicativ

Problema C.1.